Abstract viewpoint on Schönberg correspondence: the Haagerup property via approximating semigroups and their generators

based on joint work with M. Caspers, M. Daws, P. Fima and S. White

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Schönberg meets Haagerup
Schönberg and Haagerup for classical groups

Schönberg correspondence

\( G \) – discrete group.
\( \varphi : G \to \mathbb{C} \) is *positive definite* if \( \varphi(e) = 1 \) and for all \( n \in \mathbb{N}, \ g_1, \ldots, g_n \in G, \lambda_1, \ldots, \lambda_n \in \mathbb{C} \)

\[
\sum_{i,j=1}^{n} \varphi(g_i^{-1}g_j) \lambda_i \lambda_j \geq 0.
\]

\( \psi : G \to \mathbb{C} \) is *conditionally positive definite* if \( \psi(e) = 0 \) and for all \( n \in \mathbb{N}, \ g_1, \ldots, g_n \in G, \lambda_1, \ldots, \lambda_n \in \mathbb{C} \)

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\sum_{i=1}^{n} \lambda_i = 0 \implies \sum_{i,j=1}^{n} \psi(g_i^{-1}g_j) \lambda_i \lambda_j \geq 0.
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**Theorem (Schönberg correspondence)**

A function \( \psi : G \to \mathbb{C} \) is conditionally positive definite if and only if for each \( t \geq 0 \) the function \( \varphi_t := e^{t\psi} \) is positive definite.
Schönberg meets Haagerup
Schönberg and Haagerup for classical groups

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This connects representation theory (positive definite functions) with geometry (affine actions, cocycles, etc. – conditionally positive/negative definite functions)
A discrete group $G$ has the **Haagerup property (HAP)** if the following equivalent properties hold:

1. $G$ admits a mixing unitary representation which weakly contains the trivial representation;
2. there exists a normalised sequence of positive definite functions $\varphi_n$ on $G$ vanishing at infinity convergent to 1 pointwise;
3. there exists a real, proper, conditionally positive definite function $\psi$ on $G$;
4. $G$ admits a proper affine action on a real Hilbert space.
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Equivalence (ii) $\iff$ (iii)

(iii) $\implies$ (ii) easy Schönberg correspondence
(ii) $\implies$ (iii) defining

$$\psi(g) = \sum_{n=1}^{\infty} \alpha_n (\varphi_n(g) - 1)$$

for some quickly convergent to 1 sequence of $c_0$ functions $\varphi_n$ and $\alpha_n \uparrow \infty$.

Corollary

$G$ has HAP iff it admits a continuous ‘semigroup’ $\varphi_t = \exp(t\psi)$ of positive definite functions which are in $c_0$. 
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Corollary

\(G\) has HAP iff it admits a continuous ‘semigroup’ \(\varphi_t = \exp(t\psi)\) of positive definite functions which are in \(c_0\).
$\mathbb{G}$ – a discrete quantum group, i.e.

$\ell^\infty(\mathbb{G})$ – a von Neumann algebra, which is of the form $\prod_{i \in \mathcal{I}} M_{n_i}$, equipped with the coproduct

$$\Delta : \ell^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{G}) \otimes \ell^\infty(\mathbb{G})$$

carrying all the information about $\mathbb{G}$

$c_0(\mathbb{G}) = \bigoplus_{i \in \mathcal{I}} M_{n_i}$ – the corresponding $C^*$-object

$\ell^2(\mathbb{G})$ – the GNS Hilbert space of the right invariant Haar weight on $\ell^\infty(\mathbb{G})$

$\mathbb{G}$ unimodular if the left and right weights coincide
Discrete quantum groups – general notations

\( \mathbb{G} \) – a discrete quantum group, i.e.

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\( \mathbb{G} \) **unimodular** if the left and right weights coincide
Each discrete quantum group $G$ admits the dual compact quantum group $\hat{G}$.

$L^\infty(\hat{G}), C(\hat{G})$ – subalgebras of $B(\ell^2(G))$

$L^\infty(G)$ has a canonical (Haar) normal state – tracial iff $G$ is unimodular

$C^u(\hat{G})$ – a ‘universal’ version of $C(\hat{G})$

$C^u(\hat{G})$ contains a natural dense Hopf $*$-algebra, $\text{Pol}(\hat{G})$; we have a counit $\epsilon : C^u(\hat{G}) \to \mathbb{C}$

states on $C^u(\hat{G})$ $\longleftrightarrow$ states on $\text{Pol}(\hat{G})$

In particular for $G$ – discrete group

$L^\infty(\hat{G}) = \text{VN}(G)$

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Dual (compact) quantum groups

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In particular for $G$ – discrete group

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What should positive definite functions on \( \mathbb{G} \) be? There are at least two possible points of view:

- via Bochner’s theorem, states on \( C^u(\hat{\mathbb{G}}) \) (i.e. states on ‘\( C^*(\mathbb{G}) \)’);
- elements in \( \ell^\infty(\mathbb{G}) \) yielding ‘completely positive multipliers’ on \( L^\infty(\hat{\mathbb{G}}) \) (i.e. multipliers on ‘\( VN(\mathbb{G}) \)’).

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What should positive definite functions on $G$ be? There are at least two possible points of view:

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Actually they are equivalent.
Theorem (M.Daws, P.Fima, S.White, AS)

Let $G$ be a discrete quantum group. The following conditions are equivalent (and can be used as the definition of HAP):

- $G$ admits a mixing representation weakly containing the trivial representation;
- $c_0(G)$ admits an approximate unit built of positive definite functions (equivalently, of states on $C_u(\hat{G})$);
- $G$ admits a symmetric proper generating functional (i.e. a ‘real proper conditionally positive function’);
- $G$ admits a real proper cocycle.
Cpd functions, and convolution semigroups of states

Recall: states on \( \text{Pol}(\widehat{G}) \) ↔ states on \( C^u(\widehat{G}) \). As \( \text{Pol}(\widehat{G}) = \text{Lin}\{u_{ij}^\alpha : \alpha \in \text{Irr} \widehat{G}, i, j = 1, \ldots, n_\alpha\} \),
so functionals on \( \text{Pol}(\widehat{G}) \) ↔ families of matrices in \( M_{n_\alpha} \):

\[
\mu_{i,j}^\alpha = \mu(u_{ij}^\alpha)
\]

**Definition**

A convolution semigroup of states on \( \text{Pol}(\widehat{G}) \) is a family \((\mu_t)_{t \geq 0}\) of states on \( \text{Pol}(\widehat{G}) \) such that

\[
i \quad \mu_{t+s} = \mu_t \ast \mu_s := (\mu_t \otimes \mu_s) \circ \Delta_{\widehat{G}}, \quad t, s \geq 0;
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\[
ii \quad \mu_t(a) \xrightarrow{t \to 0^+} \mu_0(a) := \epsilon(a), \quad a \in \text{Pol}(\widehat{G}).
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Convolution of functionals corresponds to multiplying matrices – for each \( \alpha \in \text{Irr} \widehat{G} \) there is a pointwise continuous semigroup of matrices \((\mu^\alpha_t)\).
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The following theorem is essentially due to M. Schürmann.

**Theorem (Quantum Schönberg correspondence)**

Each convolution semigroup of states on $\text{Pol}(\hat{G})$ possesses a generating functional $L : \text{Pol}(\hat{G}) \rightarrow \mathbb{C}$:

$$L(a) = \lim_{t \to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \quad a \in \text{Pol}(\hat{G}).$$

The functional $L$ is selfadjoint, vanishes at 1 and is positive on the kernel of the counit; in turn each functional enjoying these properties generates a convolution semigroup of states.

Thus – conditionally positive definite functions on $G$ correspond to generating functionals on $\text{Pol}(\hat{G})$. 
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**Proof.**

One direction is just differentiating $\mu_t$ at 0 (note that the derivative exists!). The other is again a GNS construction + an application of quantum stochastic processes of Hudson and Parthasarathy.
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Theorem (DFWS)

Let $\mathbb{G}$ be a discrete quantum group. The following are equivalent:

1. $c_0(\mathbb{G})$ admits an approximate unit built of positive definite functions;
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Here **symmetric** means that each matrix $L^\alpha \in M_{n_\alpha}$, $\alpha \in \text{Irr}\mathbb{G}$, is self-adjoint, and **proper** that for $\alpha \to \infty$ we have $L^\alpha \searrow -\infty$. 
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$(i) \implies (ii)$ first show that the approximate unit built of ‘positive definite $c_0$ functions’ on $G$ (i.e. suitable states on $\text{Pol}(\hat{G})$) can be chosen invariant under both the scaling automorphism group and the unitary antipode $R$ (i.e. $\mu_n^\alpha$ are self-adjoint matrices). Then repeat the classical proof.
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Semigroup corollary

**Theorem (DFWS)**

Let $G$ be a discrete quantum group. The following are equivalent:

i. $G$ has HAP (i.e. $c_0(G)$ admits an approximate unit built of positive definite functions);

ii. $\hat{G}$ admits a symmetric proper generating functional.

**Corollary**

$G$ has HAP if and only if there exists a convolution semigroup of states $(\mu_t)_{t \geq 0}$ on $\text{Pol}(\hat{G})$ built of ‘$c_0$-functions’.

When $G$ is *unimodular*, one can make in addition the matrices $L^\alpha$ scalar (a centrality property).
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Aside on Property (T) for discrete quantum groups

**Theorem (Kyed)**
A unimodular $G$ does not have Kazhdan property (T) iff it admits a generating functional $L$ which is unbounded on $\text{Pol}(\hat{G})$.

**Proposition (DSW)**
A unimodular $G$ does not have Kazhdan property (T) iff it admits a generating functional $L$ such that the matrices $L^\alpha$ are self-adjoint and the family of spectra of $L^\alpha$ is unbounded from below.

When can we add here the assumption that $L^\alpha$ is scalar? Important for understanding quantum $T^{(1,1)}$!
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A vNa $M$ with a faithful normal tracial state $\tau$ has the **von Neumann algebraic Haagerup approximation property** if there exists a net of completely positive, $\tau$-reducing, normal maps $(\Phi_i)_{i \in I}$ on $M$ such that the GNS-induced maps $T_i$ on $L^2(M, \tau)$ are compact and the net $(T_i)_{i \in I}$ converges to $I_{L^2(M,\tau)}$ strongly.

$L^2(M, \tau)$ – the GNS Hilbert space of the pair $(M, \tau)$

$$T_i(x \Omega _\tau) = \Phi_i(x) \Omega _\tau, \quad x \in M.$$ 

P. Jolissaint showed that this property does not depend on the choice of $\tau$ – so the vNa HAP is a property of a (finite) von Neumann algebra. The maps $\Phi_i$ in the definition of the vNa HAP can be chosen Markov – i.e. unital and trace preserving.
Haagerup approximation property for finite von Neumann algebras

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Schönberg meets Haagerup
HAP for tracial von Neumann algebras

Classical HAP via the approximation property for the von Neumann algebra

Theorem (Choda)

A discrete group $\Gamma$ has HAP if and only if $\text{VN}(\Gamma)$ has the von Neumann algebraic Haagerup approximation property.

Proof.

If $\Gamma$ has HAP, we have ‘good’ positive definite functions, so we can use them to construct good ‘Herz-Schur’ multipliers on $\text{VN}(\Gamma)$, which are $L^2$-compact and converge to identity pointwise $\sigma$-weakly.

The other direction is based on ‘averaging’ approximating maps on into multipliers $\text{VN}(\Gamma)$: more specifically, defining

$$\varphi(\gamma) = \tau(\Phi(\lambda_{\gamma^{-1}})\lambda_{\gamma}), \quad \gamma \in \Gamma,$$

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$$\varphi(\gamma) = \tau(\Phi(\lambda_{\gamma^{-1}})\lambda_\gamma), \quad \gamma \in \Gamma,$$

yields ‘good’ positive definite functions.
Schönberg meets Haagerup
HAP for tracial von Neumann algebras

Classical HAP via the approximation property for the von Neumann algebra

Theorem (Choda)
A discrete group $\Gamma$ has HAP if and only if $\text{VN}(\Gamma)$ has the von Neumann algebraic Haagerup approximation property.

Proof.
If $\Gamma$ has HAP, we have ‘good’ positive definite functions, so we can use them to construct good ‘Herz-Schur’ multipliers on $\text{VN}(\Gamma)$, which are $L^2$-compact and converge to identity pointwise $\sigma$-weakly. The other direction is based on ‘averaging’ approximating maps on into multipliers $\text{VN}(\Gamma)$: more specifically, defining

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Quantum group HAP via the approximation property for the vNa

Recall that if $\mathbb{G}$ is unimodular, then the Haar state of $\hat{\mathbb{G}}$ is a trace (in particular, $L^\infty(\hat{\mathbb{G}})$ is a finite von Neumann algebra).

**Theorem (DFSW, see also J.Kraus + Z.-J.Ruan)**

Let $\mathbb{G}$ be a discrete **unimodular** quantum group. Then $\mathbb{G}$ has HAP if and only if $L^\infty(\hat{\mathbb{G}})$ has the von Neumann algebraic Haagerup approximation property.

**Proof.**

Follows the classical idea of Choda: if $\mathbb{G}$ has HAP, we have good positive definite functions, so constructing multipliers out of them (see M.Junge + M.Neufang + Z.J.Ruan, later also M.Daws) yields the approximation property for $L^\infty(\hat{\mathbb{G}})$ (this does not use the unimodularity).

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What if a discrete quantum group $\mathbb{G}$ is \textbf{not} unimodular? The Haar state $h$ on $L^{\infty} (\hat{\mathbb{G}})$ is no longer tracial. But...

**Theorem**

Let $\mathbb{G}$ be a discrete quantum group with the Haagerup property. Let $M = L^{\infty} (\hat{\mathbb{G}})$. There exists a net of Markov maps $(\Phi_i)_{i \in I}$ on $M$ such that each of the respective GNS-induced maps $T_i$ on $\ell^2(\mathbb{G}) \approx L^2 (M, h)$ is compact and the net $(T_i)_{i \in I}$ converges to $I_{L^2(M,h)}$ strongly. Moreover one can choose $\Phi_i$ commuting with the action of the modular group.
Definition (M. Caspers + AS)

Let \((M, \varphi)\) be a von Neumann algebra with a \textbf{faithful normal semifinite weight}. We say that \((M, \varphi)\) has the Haagerup property if there exists a net of normal completely positive, \(\varphi\)-reducing maps \((\Phi_i)_{i \in I}\) on \(M\) such that the GNS-induced maps \(T_i\) on \(L^2(M, \varphi)\) are compact and the net \((T_i)_{i \in I}\) converges to \(I_{L^2(M, \varphi)}\) strongly.
Independence of the choice of weight

**Theorem (CS)**

The Haagerup property does not depend on the choice of a faithful normal semifinite weight.

Idea of the proof:

- show that in the state case can always make the approximations uniformly bounded;
- show that \((M, \varphi)\) has HAP iff all its ‘nice’ corners have HAP;
- prove that one can change weights if the algebra is semifinite;
- prove that HAP is stable under passing to crossed products by \((\varphi\text{-preserving})\) actions of amenable groups;
- use the Takesaki-Takai duality for the crossed products by the modular action.
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At the same time R. Okayasu and R. Tomatsu developed another approach to the Haagerup property based on the standard form of the algebra $M$ (the approximating maps in their approach act on the Hilbert space).

**Theorem (COST)**

A vNa $M$ has the Haagerup property in the sense of CS if and only if it has the Haagerup property in the sense of OT.
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A vNa $M$ has the Haagerup property in the sense of CS if and only if it has the Haagerup property in the sense of OT.
KMS-induced maps

The approach of OT is related to considering the *KMS–induced* maps.

**Definition**

Let \((M, \varphi)\) be a vNa with a faithful normal semifinite weight, \(\Phi : M \to M\) a normal completely positive, \(\varphi\)-reducing map. Its KMS-implementation on \(L^2(M, \varphi)\) is (informally!) given by the formula

\[
T^{KMS}(\Omega_{\frac{1}{2}} \varphi \times \Omega_{\frac{1}{2}} \varphi) = \Omega_{\frac{1}{2}} \varphi \Phi(x) \Omega_{\frac{1}{2}} \varphi
\]

The Haagerup property is equivalent to the KMS Haagerup property:

**Definition**

\((M, \varphi)\) has the KMS Haagerup property if there exists a net of normal completely positive, \(\varphi\)-reducing maps \((\Phi_i)_{i \in I}\) on \(M\) such that the KMS-induced maps \(T_i^{KMS}\) on \(L^2(M, \varphi)\) are compact and the net \((T_i^{KMS})_{i \in I}\) converges to \(I_{L^2(M, \varphi)}\) strongly.
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\((M, \varphi)\) has the KMS Haagerup property if there exists a net of normal completely positive, \(\varphi\)-reducing maps \((\Phi_i)_{i \in I}\) on \(M\) such that the KMS-induced maps \(T_i^{KMS}\) on \(L^2(M, \varphi)\) are compact and the net \((T_i^{KMS})_{i \in I}\) converges to \(I_{L^2(M, \varphi)}\) strongly.
The KMS– and GNS–induced maps $T^{KMS}$ and $T$ coincide if the map $\Phi$ in question commutes with the modular group. If we can find such approximations, we say that $M$ has a modular Haagerup property.
Markov property

The next result is surprisingly rather technical.

**Theorem (CS)**

Suppose that $M$ has the Haagerup property and $\varphi$ is a faithful normal state on $M$. Then one can choose the approximating (in the KMS-sense) maps to be **Markov and KMS-symmetric** (i.e. their KMS-implementations are selfadjoint operators on $L^2(M, \varphi)$).
vNa HAP via semigroups

\((M, \varphi) - vNa with a faithful normal state\)

**Definition**

A Markov semigroup \(\{\Phi_t : t \geq 0\}\) on \((M, \varphi)\) is a semigroup of Markov maps on \(M\) such that for all \(x \in M\) we have \(\Phi_t(x) \overset{t \to 0^+}{\longrightarrow} \Phi_0(x) = x\) \(\sigma\)-weakly. It is **KMS-symmetric** if each \(\Phi_t\) is KMS symmetric, and **immediately** \(L^2\)-**compact** if each of the maps \(\Phi_t^{KMS}\) with \(t > 0\) is compact.

**Theorem (CS, based on P.Jolissaint and F.Martin)**

The following are equivalent:

1. \((M, \varphi)\) has the Haagerup property;
2. there exists an immediately \(L^2\)-compact KMS-symmetric Markov semigroup \(\{\Phi_t : t \geq 0\}\) on \(M\).
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The following are equivalent:

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vNa HAP via semigroups – the ideas of the proof

This does not go directly via Schönberg – we are not building the generator from the approximations!

- start from a ‘good’ approximating sequence \((\Phi_n)_{n \in \mathbb{N}}\) and pass to the Hilbert space picture, getting \((T_n)_{n \in \mathbb{N}}\)
- use the resolvent type tricks to show that one may assume that \(T_n\) commute – write formulas of the type

\[
\theta_n := \frac{1}{n} (T_1 + \cdots + T_n), \quad \Delta_n := n(I - \theta_n), \quad R_{n,\lambda} := \lambda (\lambda I + \Delta_n)^{-1},
\]

\[
\rho_{\lambda} = \lim_{n \to \infty} R_{n,\lambda}^{-1}, \quad \tilde{T}_n = \rho_{\frac{1}{n}}
\]

- play a similar game once again to obtain a one-parameter semigroup
- go back to the vNa, checking that all the properties we want are preserved.
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- play a similar game once again to obtain a one-parameter semigroup
- go back to the vNa, checking that all the properties we want are preserved.
The following is now based on an observation of S. Neshveyev (and earlier remarks of Okayasu and Tomatsu).

**Corollary**

Let $M$ be a von Neumann algebra with HAP and let $\varphi$ be a faithful normal state on $M$. Then $(M, \varphi)$ has the **modular HAP** if and only if $\varphi$ is almost periodic.
The generators of the approximating semigroup can be described by **quantum Dirichlet forms** – a very abstract form of conditionally positive definite functions!

**Theorem**

The following are equivalent:

1. \((M, \varphi)\) has the Haagerup property;
2. \(L^2(M, \varphi)\) admits an orthonormal basis \((e_n)_{n \in \mathbb{N}}\) and a non-decreasing sequence of non-negative numbers \((\lambda_n)_{n \in \mathbb{N}}\) such that \(\lim_{n \to \infty} \lambda_n = +\infty\) and the prescription

   \[
   Q(\xi) = \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi \rangle|^2, \quad \xi \in \text{Dom } Q,
   \]

   where \(\text{Dom } Q = \{\xi \in H_\varphi : \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi \rangle|^2 < \infty\}\), defines a conservative completely Dirichlet form.
Explicit examples – $O_N^+$

$\hat{G} = O_N^+$

(based on the article of Fabio Cipriani, Uwe Franz and Anna Kula)

**Proposition**

There is an explicit decomposition $\ell^2(G)$ into finite-dimensional subspaces $H_d$ such that following formula defines a completely conservative Dirichlet form on $\ell^2(G)$, satisfying the conditions in the last theorem

$$Q(\xi) = \sum_{s=1}^{\infty} \frac{U_s'(N)}{U_s(N)} \|P_{H_d}\xi\|^2, \quad \xi \in \text{Dom } Q.$$  

$U_s(N)$ - Tchebyshev polynomials of the second kind
Explicit examples – $SU_q(2)$

\[ \hat{G} = SU_q(2) \]
(based on the article of Kenny De Commer, Amaury Freslon and Makoto Yamashita)

**Proposition**

There is an explicit decomposition $\ell^2(\hat{G})$ into finite-dimensional subspaces $H_d$ such that the following formula defines a completely conservative Dirichlet form on $\ell^2(\hat{G})$, satisfying the conditions in the last theorem

\[
Q(\xi) = \sum_{d=1}^{\infty} \frac{1}{(q - q^{-1})} \ln q A_{q,d} \| P_{H_d} \xi \|^2, \quad \xi \in \text{Dom } Q,
\]

\[
A_{q,d} := \frac{1}{q^{d+1} - q^{-(d+1)}} \left( (d + 2)(q^d - q^{-d}) + d(q^{-(d+2)} - q^{d+2}) \right)
\]
Perspectives

The classical Schönberg correspondence connects representation theory with geometry.
The same is true for

- discrete quantum groups (generating functionals $\rightarrow$ cocycles!)
- even general von Neumann algebras (Dirichlet forms $\rightarrow$ Dirac operators!)

This opens many new questions, for example related to property (T).
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References

HAP for locally compact quantum groups:


This talk:


See also:


Invitation

Graduate School Topological quantum groups
Będlewo (Poland), 28th June – 11th July 2015
http://bcc.impan.pl/15TQG/

Speakers: Teodor Banica, Michael Brannan, Martijn Caspers, Kenny De Commer, Sergey Neshveyev, Zhong–Jin Ruan, Roland Speicher, Reiji Tomatsu

Topics: Quantum groups and... Hadamard matrices, approximation properties, harmonic analysis, (ergodic) actions, categories, free combinatorics, random walks, Poisson boundaries