On the Fundamental Theorem of Calculus in the lack of local convexity

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The basics

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1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|\alpha x\| = |\alpha|\|x\|$ if $\alpha \in \mathbb{R}, x \in X$;
3. there is a constant $\kappa \geq 1$ so that for any x and y in X we have $\|x + y\| \leq \kappa(\|x\| + \|y\|)$.

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If it is possible to take $\kappa = 1$ we obtain a norm.

p-normed spaces for $p < 1$

A quasi-norm $\|\cdot\|$ on X is called p-**norm** $(0 < p \leq 1)$ if it is p-**subadditive**, that is, if

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In this case the unit ball of X is an absolutely p-convex set and X is said to be a p-**normed** space.

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Classical examples of p-Banach spaces for $0 < p < 1$ are the sequence spaces ℓ_p and the function spaces $L_p[0, 1]$.

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The spaces X and Y are **Lipschitz isomorphic** if there exists a Lipschitz bijection $f : X \to Y$ so that f^{-1} is also Lipschitz.

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For example, when $1 < p < \infty$,
$X \approx_{\text{Lip}} L_p$ (resp. $X \approx_{\text{Lip}} \ell_p$) $\Rightarrow X \approx L_p$ (resp. $X \approx \ell_p$).

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However, we don't have any positive examples:

Open Question

Are there any non-locally convex separable quasi-Banach spaces with a unique Lipschitz structure?

Differentiation as a linearizing tool

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Tamarkin's question (extended)

What are the quasi-Banach spaces X such that each Lipschitz function $f : [0, 1] \longrightarrow X$ is differentiable almost everywhere?

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Those X were called **Gelfand-Fréchet spaces** by some.

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We have $\|f(t+h) - f(t)\|_p = |h|^{1/p}$, and so

$$\left\| \frac{f(t+h) - f(t)}{h} \right\|_p = |h|^{\frac{1}{p}-1} \longrightarrow 0 \quad \text{if } |h| \longrightarrow 0.$$

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That is, f is a nonconstant Lipschitz function with zero derivative everywhere!

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Theorem (Kalton, 1981)

Suppose $X^* = \{0\}$. *Then, for every* $x \in X$ *there exists a Lipschitz function* $f : [0,1] \to X$ *such that* $f(0) = 0$, $f(1) = x$, *and* $f'(t) = 0$ *for all* $t \in [0,1]$.

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Differentiation does not seem to be the right tool to linearize Lipschitz functions mapping into quasi-Banach spaces X with $X^* = \{0\}$.

Still some hope

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Lemma

Suppose $f : [0,1] \to X$ *is Lipschitz and differentiable on* $[0,1]$ *with* $f'(t) = 0$ *a.e. Then* f *is constant on* $[0,1]$.

Clarkson's answer to Tamarkin's question

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Theorem (Clarkson, 1936)

The ℓ_p-spaces for $p \geq 1$ are Gelfand-Fréchet spaces.

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From here, appealing to Lebesgue's differentiation theorem for the Bochner integral, they deduced that f is differentiable at almost all $t \in [0,1]$ with $f'(t) = g(t)$.

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Question

Are the ℓ_p-spaces for $p < 1$ Gelfand-Fréchet spaces?

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(Even knowing that $\lim_{h\to 0} \left\| \frac{f(t+h)-f(t)}{h} - g(t) \right\|_1 = 0$ a.e., so that $g(t)$ is the weak derivative of $f(t)$ in ℓ_p at almost all t).
Roughly speaking we could say that we don't know how to differentiate quasi-Banach valued functions because we don't know how to integrate them.

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$$\left\{ f : \Omega \to X \text{ Bochner measurable } : \int_\Omega \|f\|\, d\mu < \infty \right\} := L_1(\mu, X).$$

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Theorem (Ansorena-A., 2013)

Let X be a quasi-Banach space. Suppose there exist a non-purely atomic measure space (Ω, Σ, μ) and an Orlicz function φ so that the integral operator $\mathcal{I} : \overline{\mathcal{S}(\mu, X)} \to X$ given by

$$\mathcal{I}\Big(\sum_{i=1}^{n} x_i \chi_{A_i}\Big) = \sum_{i=1}^{n} x_i \mu(A_i), \quad s = \sum_{i=1}^{n} x_i \chi_{A_i} \in \mathcal{S}(\mu, X),$$

is continuous. Then X is locally convex.

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The definition of Riemann integral extends *verbatim* for vector-valued functions mapping into a quasi-Banach space X. When X is locally convex, every continuous $f : [a, b] \to X$ is Riemann-integrable, and the corresponding integral function

$$F(t) = \int_a^t f, \qquad t \in [a, b],$$

is a primitive of f, i.e., $F'(t) = f(t)$ for all $t \in [a, b]$.

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Theorem (Mazur-Orlicz, 1948)

Suppose X is a non-locally convex quasi-Banach space. Then there exists $f : [a, b] \to X$ continuous failing to be Riemann-integrable.

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Theorem (Ansorena-A., 2012)

NO, in general. If X^* is separating, there are continuous functions $f : [0,1] \to X$ that fail to have a primitive.

Interplay between integration and differentiation

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Problem: Does the fundamental theorem of calculus hold?

If $f : [a, b] \to X$ is contiunuous and Riemann-integrable, does the integral function $F(t) = \int_a^t f$ have a derivative at every $t \in [a, b]$?

On the validity of the Fundamental Theorem of Calculus

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Theorem (Ansorena-A., 2013)

The fundamental theorem of calculus breaks down for **any** *non-locally convex quasi-Banach space X even when $F(t) = \int_a^t f$ is Lipschitz on $[a, b]$.*

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The fundamental theorem of calculus breaks down for ℓ_p, $0 < p < 1$.

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1. *The integral function F is Lipschitz on $[a, b]$,*
2. *F is differentiable on $[a, b)$, but*
3. *F fails to be left-differentiable at b.*

Re-connecting with Tamarkin's question

Open Problem

Does there exist $f : [0,1] \to X$ continuous and Riemann-integrable on $[0,1]$ whose integral function $F(t) = \int_0^t f$ is Lipschitz and fails to be differentiable on a set of positive measure?

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Note that a positive answer to this problem would solve in the negative Tamarkin's question for a given non-locally convex quasi-Banach space X (since such an X would not be a Gelfand-Fréchet space).

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$$L^1_V([0,1],X) = \left\{ f(t) = \sum_{k=1}^{\infty} f_k(t)x_k : \sum_{k=1}^{\infty} \|f_k\|_1^p \|x_k\|^p < \infty \right\},$$ where

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equipped with the p-norm

$$\|f\|_{1,V} = \inf \left\{ \left(\sum_{k=1}^{\infty} \|f_k\|_1^p \|x_k\|^p \right)^{1/p} : f(t) = \sum_{k=1}^{\infty} f_k(t)x_k \right\}.$$

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does not depend on the decomposition of $f \in L^1_V([0,1], X)$, so it is consistent to define the **Vogt-integral of f on E** as

$$\int_E f(t)\, dt = \sum_{k=1}^{\infty} x_k \int_E f_k(t)\, dt.$$

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For the moment, we know it interacts as expected with respect to differentiation:

Theorem (Ansorena-A., 2013)

If $f \in L^1_V([0,1], X)$ *then*

$$\lim_{I \to t} \frac{1}{|I|} \int_I f(u)\, du = f(t), \quad a.e. \ t \in [0,1].$$

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Can we recover a quasi-Banach valued differentiable function $F\colon [a,b] \to X$ via the Riemann integral of its derivative? That is, does the formula $F(b) - F(a) = \int_a^b F'$ hold?

Barrow's rule fails in general

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Theorem (Kalton, 1981)

Suppose that X is a quasi-Banach space with $X^* = \{0\}$. Then for every $x \in X$ there exists a continuously differentiable function $F : [a, b] \to X$ such that $F(a) = 0$, $F(b) = x$, and $F' = 0$.

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Lemma

Let X be a quasi-Banach space whose dual separates points. Let $F : [a, b] \to X$ be differentiable on $[a, b]$ with F' Riemann-integrable. Then

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Remark

To impose that F' is Riemann-integrable as hypothesis in the last result is not redundant. Indeed, we have shown that there exist continuously differentiable functions from $[0, 1]$ into a quasi-Banach space whose derivatives fail to be Riemann-integrable.

The Mean Value Property

Let $\mathcal{C}^{(1)}([a, b], X)$ be the space of all $f : [a, b] \to X$ that are differentiable at every $t \in [a, b]$ with f' continuous on $[a, b]$.

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Mean Value Property for Banach spaces

Assume that $f:X\to Y$ is Gâteaux differentiable on the interval $J=\{x_0+t(y_0-x_0): t\in[0,1]\}$ connecting x_0 with y_0. Then

$$\|f(y_0)-f(x_0)\|\le \sup_{x\in J}\|f'(x)\|\|y_0-x_0\|.$$

The Mean Value Property equates with local convexity

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The Mean Value Property in quasi-Banach spaces

Let X be a quasi-Banach space. Suppose that for some $C > 0$ every nonconstant differentiable Lipschitz function $F : [0,1] \to X$ satisfies a Mean Value property

$$\|F(y) - F(x)\| \leq C \sup_{t\in[0,1]} \|F'(x + t(y - x))\| |y - x|, \quad \forall x, y \in [0,1].$$

Then X is locally convex.

Pathologies derived from the lack of a MVP

The lack of a mean value property opens the door to the existence of functions with continuous derivative mapping into quasi-Banach spaces which are not Lipschitz!

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3. *F fails to be Lipschitz on $[0,1]$.*

That's all. MANY THANKS FOR YOUR ATTENTION!