On the Fundamental Theorem of Calculus in the lack of local convexity

F. Albiac

Joint work with J. L. Ansorena (University of La Rioja)

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1. Background and motivation
2. Differentiability of quasi-Banach valued Lipschitz functions
3. Integration in quasi-Banach spaces
4. The lack of a mean value formula and its consequences
The basics

A quasi-normed space $X$ is a locally bounded topological vector space.
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1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|\alpha x\| = |\alpha|\|x\|$ if $\alpha \in \mathbb{R}, x \in X$;
3. there is a constant $\kappa \geq 1$ so that for any $x$ and $y$ in $X$ we have $\|x + y\| \leq \kappa(\|x\| + \|y\|)$. 

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If it is possible to take $\kappa = 1$ we obtain a norm.
A quasi-norm $\| \cdot \|$ on $X$ is called \textbf{$p$-norm} ($0 < p \leq 1$) if it is \textbf{$p$-subadditive}, that is, if

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A quasi-norm $\| \cdot \|$ on $X$ is called $p$-norm $(0 < p \leq 1)$ if it is $p$-subadditive, that is, if

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In this case the unit ball of $X$ is an absolutely $p$-convex set and $X$ is said to be a $p$-normed space.
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$X$ is called a **quasi-Banach space** (also **$p$-Banach space**) if $X$ is complete for this metric.
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Classical examples of $p$-Banach spaces for $0 < p < 1$ are the sequence spaces $\ell_p$ and the function spaces $L_p[0, 1]$. 

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**Background and motivation**
- Differentiability of quasi-Banach valued Lipschitz functions
- Integration in quasi-Banach spaces
- The lack of a mean value formula and its consequences

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**$p$-Banach spaces**

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On the FTC in quasi-Banach spaces
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The spaces $X$ and $Y$ are **Lipschitz isomorphic** if there exists a Lipschitz bijection $f : X \to Y$ so that $f^{-1}$ is also Lipschitz.
Lipschitz structure of Banach spaces

Fundamental Problem

If $X$ and $Y$ are Lipschitz isomorphic separable Banach spaces, are they necessarily linearly isomorphic?
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Although the general answer to the fundamental problem remains elusive, we know of many separable Banach spaces where the Lipschitz structure determines the linear structure.

For example, when $1 < p < \infty$, $X \approx_{\text{Lip}} L_p$ (resp. $X \approx_{\text{Lip}} \ell_p$) $\Rightarrow X \approx L_p$ (resp. $X \approx \ell_p$).
Lipschitz structure of quasi-Banach spaces

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**Theorem (Kalton - A., 2009)**

*The Lipschitz structure of a separable quasi-Banach space does not determine, in general, its linear structure.*
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However, we don’t have any positive examples:

**Open Question**

Are there any non-locally convex separable quasi-Banach spaces with a unique Lipschitz structure?
Differentiation as a linearizing tool

Lipschitz maps between Banach spaces are "smooth" in many cases, which makes differentiation a crucial tool to obtain linear embeddings of one space into another from Lipschitz embeddings.
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This leads naturally to wonder whether Lipschitz functions from the real line into a quasi-Banach space are differentiable:
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Tamarkin’s question (extended)
What are the quasi-Banach spaces $X$ such that each Lipschitz function $f : [0, 1] \to X$ is differentiable almost everywhere?
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What are the quasi-Banach spaces $X$ such that each Lipschitz function $f : [0, 1] \rightarrow X$ is differentiable almost everywhere?

Those $X$ were called **Gelfand-Fréchet spaces** by some.
Initial setbacks

Example.

Take $X = L^p[0,1]$ for $p < 1$ with the standard quasi-norm:

$$\|x\|_p = \left( \int_0^1 |x(s)|^p \, ds \right)^{1/p},$$

and consider the map $f: [0,1] \to L^p[0,1], t \to f(t) = \chi_{[0,t]}$.

We have

$$\|f(t+h) - f(t)\|_p = |h|^{1/p},$$

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$$\frac{\|f(t+h) - f(t)\|_p}{|h|^{1/p}} \to 0 \quad \text{as} \quad |h| \to 0.$$
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We have $\|f(t + h) - f(t)\|_p = |h|^{1/p}$, and so

$$\left\| \frac{f(t + h) - f(t)}{h} \right\|_p = |h|^{\frac{1}{p} - 1} \to 0 \quad \text{if} \quad |h| \to 0.$$
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That is, $f$ is a nonconstant Lipschitz function with zero derivative everywhere!
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**Theorem (Kalton, 1981)**

Suppose $X^* = \{0\}$. Then, for every $x \in X$ there exists a Lipschitz function $f : [0, 1] \to X$ such that $f(0) = 0$, $f(1) = x$, and $f'(t) = 0$ for all $t \in [0, 1]$. 

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Differentiation does not seem to be the right tool to linearize Lipschitz functions mapping into quasi-Banach spaces $X$ with $X^* = \{0\}$. 
Still some hope

On the other hand, in quasi-Banach spaces $X$ with separating dual (like the $\ell_p$-spaces for $p < 1$) there is still some initial hope thanks to the following elementary lemma.

**Lemma**
Suppose $f : [0, 1] \to X$ is Lipschitz and differentiable on $[0, 1]$ with $f'(t) = 0$ a.e. Then $f$ is constant on $[0, 1]$. 

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To address the question whether we can differentiate Lipschitz maps $f : [0, 1] \to \ell_p$ when $0 < p < 1$, let us see what happens with their locally convex relatives.
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**Theorem (Clarkson, 1936)**

*The $\ell_p$-spaces for $p \geq 1$ are Gelfand-Fréchet spaces.*
Sketch of Proof (Dunford-Morse):

Let \( f : [0,1] \to \ell^p \), \( t \mapsto \sum a_n(t) e_n \), be a Lipschitz map. By composing with the coordinate functionals \( e^*_n : \ell^p \to \mathbb{R} \), each \( a_n : [0,1] \to \mathbb{R} \) is Lipschitz, hence differentiable a.e. \( t \in [0,1] \).

The fact that \((e_n)\) is boundedly complete yields that the series \( \sum a'_n(t) e_n \) converges to some \( g(t) \in \ell^p \) almost everywhere.

Using the recently invented Bochner integral, for every \( t \in [0,1] \):

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 f(t) = \sum \left( \int_0^t a'_n(s) \, ds \right) e_n = \int_0^t \sum a'_n(s) e_n \, ds = \int_0^t g(s) \, ds.
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From here, appealing to Lebesgue's differentiation theorem for the Bochner integral, they deduced that \( f \) is differentiable at almost all \( t \in [0,1] \) with \( f'(t) = g(t) \).
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**Question**

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Roughly speaking we could say that we don’t know how to differentiate quasi-Banach valued functions because we don’t know how to integrate them.
Why does Bocher-integration fail in quasi-Banach spaces?

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$$\left\{ f : \Omega \to X \text{ Bochner measurable} : \int_{\Omega} \|f\| \, d\mu < \infty \right\} := L_1(\mu, X).$$
However, when we try to mimic the above construction in quasi-Banach spaces $X$, we discover that local convexity is not only a sufficient condition but it is also necessary:
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**Theorem (Ansorena-A., 2013)**

Let $X$ be a quasi-Banach space. Suppose there exist a non-purely atomic measure space $(\Omega, \Sigma, \mu)$ and an Orlicz function $\varphi$ so that the integral operator $I : S(\mu, X) \to X$ given by

$$I\left(\sum_{i=1}^{n} x_i \chi_{A_i}\right) = \sum_{i=1}^{n} x_i \mu(A_i), \quad s = \sum_{i=1}^{n} x_i \chi_{A_i} \in S(\mu, X),$$

is continuous. Then $X$ is locally convex.
What about Riemann-integration?

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The definition of Riemann integral extends *verbatim* for vector-valued functions mapping into a quasi-Banach space $X$. When $X$ is locally convex, every continuous $f : [a, b] \to X$ is Riemann-integrable, and the corresponding integral function

$$F(t) = \int_a^t f, \quad t \in [a, b],$$

is a primitive of $f$, i.e., $F'(t) = f(t)$ for all $t \in [a, b]$. 
However, the situation changes dramatically if local convexity is lifted:
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**Theorem (Mazur-Orlicz, 1948)**

Suppose $X$ is a non-locally convex quasi-Banach space. Then there exists $f : [a, b] \rightarrow X$ continuous failing to be Riemann-integrable.
Existence of Primitives for continuous functions into quasi-Banach spaces

Since, in view of the above, the natural way to obtain primitives may fail, it is natural to ask:
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**Question (M.M. Popov, Studia 1994)**

*Does every continuous function from a compact interval of the real line into a given quasi-Banach space $X$ have a primitive?*
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Existence of Primitives for continuous functions into quasi-Banach spaces

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*Yes if $X^* = \{0\}$ (e.g., when $X = L_p$ for $p < 1$).*


*NO, in general. If $X^*$ is separating, there are continuous functions $f : [0, 1] \rightarrow X$ that fail to have a primitive.*
Suppose $X$ is a non-locally convex quasi-Banach space.
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$$F(t) = \int_a^t f(u)du, \quad t \in [a, b],$$

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is not a trivial question.

**Problem:** Does the fundamental theorem of calculus hold?

*If $f : [a, b] \rightarrow X$ is continuous and Riemann-integrable, does the integral function $F(t) = \int_a^t f$ have a derivative at every $t \in [a, b]$?*
On the validity of the Fundamental Theorem of Calculus

Theorem (M. M. Popov., 1994)

The fundamental theorem of calculus breaks down for $\ell_p$, $0 < p < 1$. 
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The fundamental theorem of calculus breaks down for any non-locally convex quasi-Banach space $X$ even when $F(t) = \int_a^t f$ is Lipschitz on $[a, b]$. 
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The fundamental theorem of calculus breaks down for any non-locally convex quasi-Banach space $X$ even when $F(t) = \int_a^t f$ is Lipschitz on $[a, b]$. To be precise, there exists $f : [a, b] \to X$ continuous and Riemann-integrable so that:

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1. The integral function \( F \) is Lipschitz on \([a, b] \),
2. \( F \) is differentiable on \([a, b) \), but
3. \( F \) fails to be left-differentiable at \( b \).
Re-connecting with Tamarkin’s question

Open Problem

Does there exist $f : [0, 1] \rightarrow X$ continuous and Riemann-integrable on $[0, 1]$ whose integral function $F(t) = \int_0^t f$ is Lipschitz and fails to be differentiable on a set of positive measure?
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Open Problem

Does there exist $f : [0, 1] \rightarrow X$ continuous and Riemann-integrable on $[0, 1]$ whose integral function $F(t) = \int_0^t f$ is Lipschitz and fails to be differentiable on a set of positive measure?

Note that a positive answer to this problem would solve in the negative Tamarkin’s question for a given non-locally convex quasi-Banach space $X$ (since such an $X$ would not be a Gelfand-Fréchet space).
An integral designed for $p$-Banach spaces

In 1967, Vogt introduced a concept of integrability for $p$-Banach spaces ($0 < p < 1$) which tried to replace Bochner’s integral.
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An integral designed for $p$-Banach spaces

In 1967, Vogt introduced a concept of integrability for $p$-Banach spaces ($0 < p < 1$) which tried to replace Bochner’s integral. Let $X$ be $p$-Banach. The space of $X$-valued Vogt-integrable functions on the interval $[0, 1]$ is the space

$$L^1_V([0, 1], X) = \left\{ f(t) = \sum_{k=1}^{\infty} f_k(t)x_k : \sum_{k=1}^{\infty} \left\| f_k \right\|_1^p \left\| x_k \right\|_p^p < \infty \right\},$$

where $f_k \in L^1[0, 1]$ and $x_k \in X$ for all $k$. 

F. Albiac

On the FTC in quasi-Banach spaces
An integral designed for $p$-Banach spaces

In 1967, Vogt introduced a concept of integrability for $p$-Banach spaces ($0 < p < 1$) which tried to replace Bochner’s integral. Let $X$ be $p$-Banach. The space of $X$-valued Vogt-integrable functions on the interval $[0, 1]$ is the space

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where $f_k \in L_1[0, 1]$ and $x_k \in X$ for all $k$,

equipped with the $p$-norm

$$\|f\|_{1,V} = \inf \left\{ \left( \sum_{k=1}^{\infty} \|f_k\|_1^p \|x_k\|^p \right)^{1/p} : f(t) = \sum_{k=1}^{\infty} f_k(t)x_k \right\}.$$
For $E$ measurable, the expression

$$
\sum_{k=1}^{\infty} x_k \int_{E} f_k(t) \, dt
$$

does not depend on the decomposition of $f \in L^1_V([0, 1], X)$,
For $E$ measurable, the expression

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does not depend on the decomposition of $f \in L^1_V([0, 1], X)$, so it is consistent to define the Vogt-integral of $f$ on $E$ as

$$\int_E f(t) \, dt = \sum_{k=1}^{\infty} x_k \int_E f_k(t) \, dt.$$
The applicability of this tool was not properly investigated when it was introduced.
The FTC for the Vogt Integral

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**Theorem (Ansorena-A., 2013)**

If \( f \in L^1_V([0, 1], X) \) then

\[
\lim_{l \to t} \frac{1}{|l|} \int_I f(u) \, du = f(t), \text{ a.e. } t \in [0, 1].
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Let us concentrate now on the converse problem:
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Can we recover a quasi-Banach valued differentiable function $F : [a, b] \to X$ via the Riemann integral of its derivative?
Integration of derivatives

Let us concentrate now on the converse problem:

Can we recover a quasi-Banach valued differentiable function $F : [a, b] \rightarrow X$ via the Riemann integral of its derivative? That is, does the formula $F(b) - F(a) = \int_a^b F'$ hold?
Barrow’s rule fails in general

Clearly, Barrow’s rule breaks down, in general, for quasi-Banach spaces if we take into account (once more) the following theorem of Kalton:
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Clearly, Barrow’s rule breaks down, in general, for quasi-Banach spaces if we take into account (once more) the following theorem of Kalton:

Theorem (Kalton, 1981)

Suppose that $X$ is a quasi-Banach space with $X^* = \{0\}$. Then for every $x \in X$ there exists a continuously differentiable function $F : [a, b] \to X$ such that $F(a) = 0$, $F(b) = x$, and $F' = 0$. 
The situation is different in quasi-Banach spaces with rich dual:
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**Lemma**

Let $X$ be a quasi-Banach space whose dual separates points. Let $F : [a, b] \rightarrow X$ be differentiable on $[a, b]$ with $F'$ Riemann-integrable. Then

$$\int_a^b F' = F(b) - F(a).$$
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Let $X$ be a quasi-Banach space whose dual separates points. Let $F : [a, b] \to X$ be differentiable on $[a, b]$ with $F'$ Riemann-integrable. Then

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**Remark**

To impose that $F'$ is Riemann-integrable as hypothesis in the last result is not redundant.
The situation is different in quasi-Banach spaces with rich dual:

**Lemma**

Let $X$ be a quasi-Banach space whose dual separates points. Let $F : [a, b] \to X$ be differentiable on $[a, b]$ with $F'$ Riemann-integrable. Then

$$\int_a^b F' = F(b) - F(a).$$

**Remark**

To impose that $F'$ is Riemann-integrable as hypothesis in the last result is not redundant. Indeed, we have shown that there exist continuously differentiable functions from $[0, 1]$ into a quasi-Banach space whose derivatives fail to be Riemann-integrable.
The Mean Value Property

Let $C^{(1)}([a, b], X)$ be the space of all $f : [a, b] \to X$ that are differentiable at every $t \in [a, b]$ with $f'$ continuous on $[a, b]$. 
The Mean Value Property

Let \( C^{(1)}([a, b], X) \) be the space of all \( f : [a, b] \rightarrow X \) that are differentiable at every \( t \in [a, b] \) with \( f' \) continuous on \([a, b]\).

When \( X \) is a Banach space, a function \( f \in C^{(1)}([a, b], X) \) is Lipschitz on \([a, b]\) thanks to the mean value property.
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When $X$ is a Banach space, a function $f \in C^{(1)}([a, b], X)$ is Lipschitz on $[a, b]$ thanks to the mean value property.

Mean Value Property for Banach spaces

Assume that $f : X \rightarrow Y$ is Gâteaux differentiable on the interval $J = \{x_0 + t(y_0 - x_0) : t \in [0, 1]\}$ connecting $x_0$ with $y_0$. Then

$$
\|f(y_0) - f(x_0)\| \leq \sup_{x \in J} \|f'(x)\| \|y_0 - x_0\|.
$$
The Mean Value Property equates with local convexity

As it happens, the MVP characterizes local convexity!
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The Mean Value Property in quasi-Banach spaces

Let $X$ be a quasi-Banach space. Suppose that for some $C > 0$ every nonconstant differentiable Lipschitz function $F : [0, 1] \to X$ satisfies a Mean Value property

$$\| F(y) - F(x) \| \leq C \sup_{t \in [0,1]} \| F'(x + t(y - x)) \| |y - x|, \quad \forall x, y \in [0, 1].$$

Then $X$ is locally convex.
Pathologies derived from the lack of a MVP

The lack of a mean value property opens the door to the existence of functions with continuous derivative mapping into quasi-Banach spaces which are not Lipschitz!
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Let $X$ be a non-locally convex quasi-Banach space. Then, there exists $F : [0, 1] \rightarrow X$ such that

$F$ fails to be Lipschitz on $[0, 1]$. 

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On the FTC in quasi-Banach spaces
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Pathologies derived from the lack of a MVP

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1. $F$ is differentiable on $[0, 1]$,
2. $F'$ is continuous and Riemann-integrable on $[0, 1]$ and $F(t) = \int_0^1 F'(u) \, du$ for all $t \in [0, 1]$,
3. $F$ fails to be Lipschitz on $[0, 1]$. 
That's all. **Many thanks for your attention!**