Greedy Bases and the Greedy Constant

October, 2014

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Greedy Approximants

Let $(e_i)_{i=1}^{\infty}$ be a seminormalized basis for a Banach space X, i.e. $\exists 0 < a \leqslant b$

$$a \leqslant \|e_i\| \leqslant b \qquad (i \geqslant 1).$$

For $x \in X$:

$$x = \sum_{i=1}^{\infty} a_i e_i \qquad (a_i = e_i^*(x)).$$

Define a set Λ_m of m coefficient indices:

$$|\Lambda_m| = m; \qquad \min\{|a_i| : i \in \Lambda_m\} \geqslant \max\{|a_i| : i \notin \Lambda_m\}.$$

Then

$$G_m(x) = P_{\Lambda_m}(x) := \sum_{i \in \Lambda_m} a_i e_i$$

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$$G_m(x) = P_{\Lambda_m}(x) := \sum_{i \in \Lambda_m} a_i e_i$$

is an *m-th greedy approximation* to x.

The Thresholding Greedy Algorithm (TGA) converges, if $G_m(x) \to x$.

Example

Suppose $x = e_1 - 3e_2 - 4e_5 + 3e_7$.

$$G_1(x) = -4e_5$$

$$G_2(x) = -4e_5 - 3e_2 \quad \text{or} \quad G_2(x) = -4e_5 + 3e_7$$

$$G_3(x) = -4e_5 - 3e_2 + 3e_7,$$

$$G_4(x) = -4e_5 - 3e_2 + 3e_7 + e_1$$

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Convergence of the TGA

Definition (Konyagin-Temlyakov, 1999)

(e_i) is quasi-greedy (QG) if there exists a constant K (the quasi-greedy constant) such that

$$\|G_n(x)\| \leqslant K\|x\| \quad (x \in X, n \geqslant 1).$$

Theorem (Wojtaszczyk, 2000)

(e_i) is quasi-greedy (QG) if and only if the TGA converges, i.e. $\forall x = \sum_{i=1}^{\infty} a_i e_i$,

$$x = \sum_{i=1}^{\infty} a_{\rho_x(i)}(x) e_{\rho_x(j)}$$

for every greedy ordering $\rho_x : \mathbb{N} \to \mathbb{N}$,

$$|a_{\rho_x(i)}| \geqslant |a_{\rho_x(i+1)}| \qquad (i \geqslant 1).$$

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QG Bases

- (e_i) is unconditional if every rearrangement of $x = \sum e_i^*(x) e_i$ converges, so

$$\text{unconditional} \quad \Rightarrow \quad \text{QG}.$$

- (Wojtaszczyk) ℓ_2 has a conditional QG basis
- (DKK) Every QG basis of c_0 is unconditional
- (DKK) $L_1[0,1]$ has a QG basis (but not the Haar system)
- (DKK) $C[0,1]$ does not have any QG basis

Question

Does every Banach space contain a QG basic sequence

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Best n-term approximation

For $x \in X$, $\sigma_n(x)$ is the error in the best n-term approximation to x:

$$\sigma_n(x) = \inf\{\|x - \sum_{j \in A} \alpha_j e_j\| : \ |A| = n, \ \alpha_j \in \mathbf{R}\}.$$

Hence

$$\sigma_n(x) \leqslant \|x - G_n(x)\|.$$

Definition

(e_i) is greedy with greedy constant $C \geqslant 1$ if

$$\|x - G_n(x)\| \leqslant C\sigma_n(x) \qquad (x \in X, n \in \mathbf{N}).$$

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Example

The unit vector basis of ℓ_p or c_0 is 1-greedy, i.e.

$$\|x - G_n(x)\| = \sigma_n(x).$$

Theorem A (Temlyakov, 1998)

For $d \geqslant 1$ the multivariate Haar basis of $L_p[0,1]^d$ (normalized in $L_p[0,1]^d$) is C_p-greedy for $1 < p < \infty$.

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Democratic Bases

Definition

(e_i) is democratic with constant Δ (Δ-democratic) if $\forall$ finite $A, B \subset \mathbb{N}$,

$$|A| \leqslant |B| \Rightarrow \|\sum_{i \in A} e_i\| \leqslant \Delta \|\sum_{i \in B} e_i\|.$$

$$|A| = |B| \Rightarrow \frac{1}{\Delta}\|\sum_{i \in B} e_i\| \leqslant \|\sum_{i \in A} e_i\| \leqslant \Delta \|\sum_{i \in B} e_i\|.$$

Recall that a basis is subsymmetric if it is unconditional and equivalent to its subsequences:

subsymmetric $\Rightarrow$ democratic & unconditional.

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Characterization of Greedy Bases

Theorem (Konyagin-Temlyakov, 1999)

Greedy $\Leftrightarrow$ unconditional & democratic.

Example

Subsymmetric bases are greedy

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The Haar basis of $L_p[0,1]$ (normalized in $L_p[0,1]$) is C_p-greedy for $1 < p < \infty$.

Remark

The greedy constant $C_p > 1$ unless $p = 2$.

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K-Greedy $\Rightarrow$ K-Democratic

Suppose $|A| \leqslant |B| := n$. Let $\varepsilon > 0$. Consider

$$x = (1+\varepsilon) \sum_{i \in B \setminus A} e_i + \sum_{i \in A} e_i.$$

For $k = |B \setminus A|$, $G_k(x) = (1+\varepsilon)\sum_{i \in B \setminus A} e_i$.

$$\begin{aligned} \|\sum_{i \in A} e_i\| &= \|x - G_k(x)\| \\ &\leqslant K\sigma_k(x) \\ &\leqslant K\|x - \sum_{i \in A \setminus B} e_i\| \\ &= K\|\sum_{i \in A \cap B} e_i + (1+\varepsilon)\sum_{i \in B \setminus A} e_i\| \end{aligned}$$

since $|A \setminus B| \leqslant k$. Let $\varepsilon \to 0$:

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Almost greedy bases

Definition

The error in the best *n* term projection approximating *x* is given by

$$\tilde{\sigma}_n(x) = \inf\{\|x - \sum_{j\in A} e_j^*(x)e_j\| : \ |A| \leqslant n\}.$$

Theorem (DKKT)

The following are equivalent:

- $\exists C$ *such that*

$$\|x - G_n(x)\| \leqslant C\tilde{\sigma}_n(x) \qquad (x \in X, n \geqslant 1).$$

- (e_i) *is QG and democratic.*
- $\exists C$ *such that*

$$\|x - G_{2n}(x)\| \leqslant C\sigma_n(x) \qquad (x \in X, n \geqslant 1).$$

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(e_i) is almost greedy if (e_i) is QG and democratic

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Suppose X has a basis and contains a complemented subspace with a symmetric basis and finite cotype. Then X has a an almost greedy basis.

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Chebyshev approximation

For $x \in X$, recall

$$G_n(x) = \sum_{i \in \Lambda_n(x)} e_i^*(x) e_i.$$

Let

$$G_n^C(x) = \sum_{i \in \Lambda_n(x)} b_i e_i.$$

be a best approximation to x from $\text{span}\{e_i : i \in \Lambda_n(x)\}$.

Theorem (DKK)

Let (e_i) be almost greedy. Then, for all $x \in X$,

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where K depends on the QG and democratic constant of (e_i).

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Duality

Duality fails in general:

- If (e_i) is greedy then (e_i^*) may fail to be democratic
- If (e_i) is QG then (e_i^*) may fail to be QG.

Definition

The fundamental function $\varphi : \mathbb{N} \to \mathbb{R}$ of (e_i) is defined by:

$$\varphi(n) := \sup_{|A| \leqslant n} \|\sum_{i \in A} e_i\|.$$

Hence (e_i) is Δ-democratic if $\forall$ finite $A \subset \mathbb{N}$,

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$$\varphi(|A|) \leqslant \Delta \| \sum_{i \in A} e_i \|.$$

Definition

A fundamental function $(\varphi(n))$ has the upper regularity property (URP) if $\exists C > 0$ and $0 < \beta < 1$ such that

$$\varphi(m) \leqslant C(m/n)^{\beta}\varphi(n) \qquad (m > n).$$

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If (e_n) is a greedy (resp. almost greedy) basis whose fundamental function has URP, then (e_n^*) is a greedy (resp. almost greedy) basic sequence.

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Let (e_i) be a QG basis for a separable Hilbert space. Then both (e_i) and (e_i^*) are almost greedy bases for H.

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Characterization of duality

Definition

Let (e_n) be a basis for X with fundamental function (φ_n). Let (φ_n^*) be the fundamental function for (e_n^*). Then (e_n) is *C*-bidemocratic if

$$\varphi(n)\varphi^*(n) \leqslant Cn \qquad (n \in \mathbb{N}),$$

i.e.,

$$\left\| \frac{\varphi(|A|)}{|A|} \sum_{i \in A} e_i^* \right\| \leqslant C \qquad (A \subset \mathbb{N}).$$

Theorem

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Corollary

Every greedy basis with fundamental function $(\varphi(n))$ *is* *bidemocratic* *if and only if* $(\varphi(n))$ *has URP.*

1-Greedy Renormings

Definition

(e_i) is suppression C-unconditional if $\forall$ finite $A \subset \mathbb{N}$, and $\forall x = \sum a_i e_i$,

$$\|P_A(x)\| \leqslant C\|x\|.$$

Theorem

(Konyagin-Temlyakov)

- *C-unconditional and Δ-democratic $\Rightarrow$ $(C + C^3\Delta)$-greedy.*
- *K-greedy $\Rightarrow$ K-democratic & suppression K-unconditional.*

Corollary

- *1-unconditional & 1-democratic $\Rightarrow$ 2-greedy*
- *1-greedy $\Rightarrow$ suppression 1-unconditional & 1-democratic*

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Theorem (DOSZ)

- $\exists$ *a* 1*-unconditional &* 1*-democratic basis that is not* (2 $-\varepsilon$)*-greedy for any* $\varepsilon > 0$*.*
- $\exists$ *a* 1*-greedy basis that is not* (2 $-\varepsilon$)*-unconditional for any* $\varepsilon > 0$*.*

Theorem (Albiac-Wojtaszczyk)

A suppression 1*-unconditional basis is* 1*-greedy iff* $\|x\|$ *is invariant under all greedy permutations* $\Pi(x)$ *of* x*.*

Example

A greedy permutation of x moves to other coordinates some of the largest coefficients, possibly changes their sign, and leaves all other nonzero coefficients unchanged. Consider

$$x = 2e_1 - 5e_2 - 4e_3 + 5e_6 - 5e_8 - e_9.$$

The vector y below is a greedy permutation of x:

$$y = 2e_1 - 4e_3 + 5e_4 + 5e_6 - 5e_7 - e_9.$$

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Corollary (Albiac-Wojtaszczyk)

- 1*-symmetric* $\Rightarrow$ 1*-greedy*
- $\exists$ *a* 1*-greedy basis that is not* 1*-symmetric*

Question (Albiac-Wojtaszczyk)

Suppose (e_i) is greedy. Is there a renorming of X so that (e_i) is 1-greedy in the new norm?

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Suppose (e_i) is suppression 1-unconditional, 1-democratic, and

$$\varphi(n) \geqslant Cn \qquad (n \geqslant 1).$$

Then (e_i) is equivalent to the unit vector basis of ℓ_1.

Corollary

The Haar basis for the dyadic Hardy space H_1 is greedy but not 1-democratic and suppression 1-unconditional (hence not 1-greedy) in any equivalent norm.

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Recent Positive Results

Theorem (DKOSZ, in press)

Suppose (e_i) is unconditional and bidemocratic. Given $\varepsilon > 0$ $\exists$ an equivalent norm so that (e_i) is 1-unconditional, 1-bidemocratic, and $(1+\varepsilon)$-greedy.

Sketch Proof.

- Characterize K-greedy bases by generalizing the Albiac-Wojtaszczyk characterization of 1-greedy bases.
- Define the new norm explicitly.
- Show the new norm is an equivalent norm using the bidemocratic property:

$$\left\| \frac{\varphi(|A|)}{|A|} \sum_{i \in A} e_i^* \right\| \leqslant C \qquad (A \subset \mathbb{N}).$$

- Show the norm satisfies the characterization for $K = 1 + \varepsilon$.

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Corollary

for $1 < p < \infty$ *and* $\varepsilon > 0$ $\exists$ *an equivalent norm on* $L_p[0,1]$ *so that the Haar basis is* $(1+\varepsilon)$*-greedy.*

Theorem (DKOSZ)

Let (e_i) *be a greedy basis. Given* $\varepsilon > 0$, $\exists$ *an equivalent norm so that* (e_i) *is* 1*-unconditional and* $(1+\varepsilon)$*-democratic, hence* $(2+\varepsilon)$*-greedy.*

The main ingredient is a combinatorial lemma which says that all democratic bases are "sufficiently bidemocratic".

Lemma

Let (e_i) *be a normalized* 1*-unconditional,* Δ*-democratic basis with fundamental function* $(\varphi(n))$*. Given* $0 < q < 1$ $\exists C(q,\Delta)$ *such that for all finite* $E \subset \mathbb{N}$ $\exists A \subset E$ *with* $|A| \geqslant q|E|$ *such that*

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Theorem

Suppose (e_i) *is a greedy basis for* X *and* $\varphi(n) \asymp n$ *(e.g. dyadic* H_1 *and Tsirelson space). Then* $\forall \varepsilon > 0$ *there is a renorming so that* (e_i) *is* $(1+\varepsilon)$*-greedy.*

Question (Albiac-Wojtaszczyk)

Is every 1-greedy basis C-symmetric?

Theorem (DOSZ3)

There is a renorming of $\ell_2 \oplus \ell_{2,1}$ for which the natural basis is 1-greedy. This basis is not subsymmetric

Remark

$\ell_{2,1}$ is a Lorentz space:

$$\|\sum a_i e_i\|_{2,1} = \sum a_i^* (\sqrt{i} - \sqrt{i-1}).$$

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Greedification

Definition

Let (e_i) be a basis for $(X, \|\cdot\|)$. For $x \in X$

- $$f(x) = \inf\{\|y\| : y \text{ is a greedy rearrangement of } x\}.$$

- $$\|x\|_1 = \inf\{\sum_{i=1}^{n} f(x_i) : x = \sum_{i=1}^{n} x_i\}.$$

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$\|\cdot\|_1 = \|\cdot\| \Leftrightarrow (e_i)$ is 1-greedy for $\|\cdot\|$.

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Open questions about the greedy constant

Question
Is every greedy basis $(1+\varepsilon)$-greedy in an equivalent norm?

Question
Is every bidemocratic greedy basis 1-greedy in an equivalent norm?

Question
Let $1 < p < \infty$. Is the Haar basis for $L_p[0,1]$ 1-greedy in an equivalent norm?