University of Zurich[UZH]

Periodically stationary multivariate autoregressive models

An applied perspective for infectious disease surveillance data

work in progress – *Multivariate Count Analysis*, Besançon, 5-7-2018

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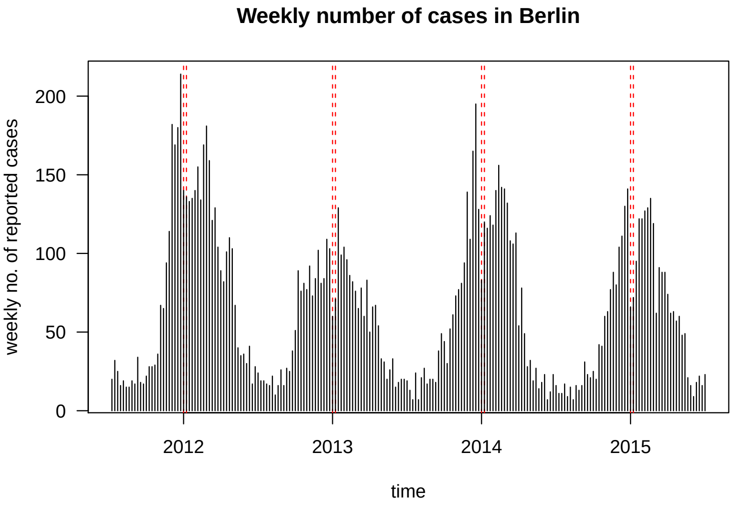
5 July 2018

Agenda

1. Infectious disease surveillance data
2. The endemic-epidemic model class
3. Adding higher-order lags
4. Periodic stationarity properties
5. Use in practical data analysis

Routine surveillance data

Routine surveillance data: norovirus (weekly)



Source: Robert Koch Institut Berlin, *survstat@rki*

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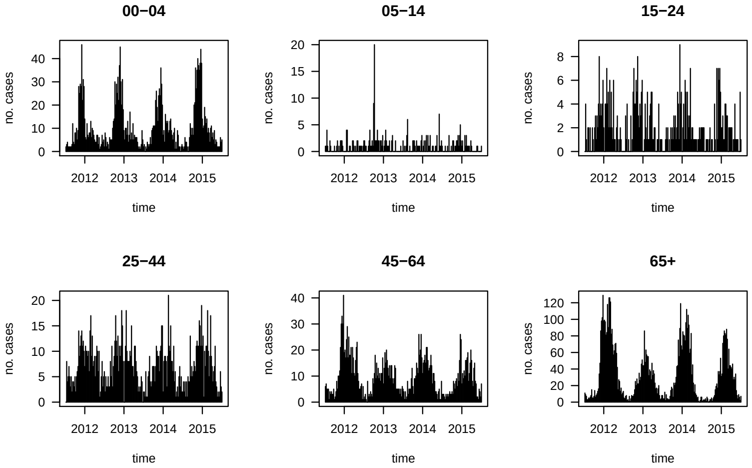

Data as used in Held, Meyer, and Bracher (2017)

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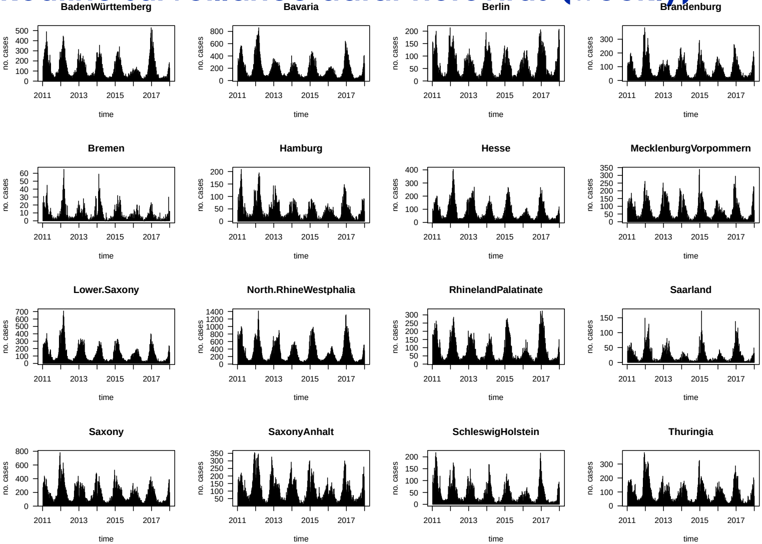
Routine surveillance data: norovirus (weekly)



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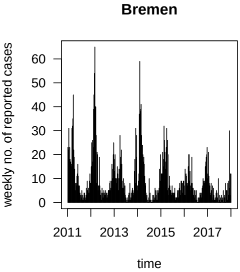
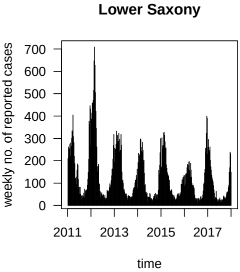




Source: Robert Koch Institut Berlin, *survstat@rki*

The endemic-epidemic model

The endemic-epidemic model (Held et al 2006)
The univariate case

For the weekly numbers of new cases we assume a process $\{X_t, t \in \mathbb{Z}\}$ with

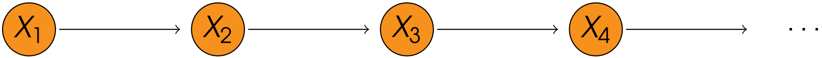
$$X_t \mid \mathcal{F}_{t-1} \sim \text{NegBin}(\lambda_t, \psi)$$

$$\mathbb{E}(X_t \mid \mathcal{F}_{t-1}) = \lambda_t = \underbrace{\nu_t}_{\text{“endemic”}} + \underbrace{\phi_t X_{t-1}}_{\text{“epidemic”}} .$$

$$\text{Var}(X_t \mid \mathcal{F}_{t-1}) = \lambda_t + \psi\lambda_t^2$$

$\rightarrow$ NB-INARCH(1)-type model

Simple Markov structure:



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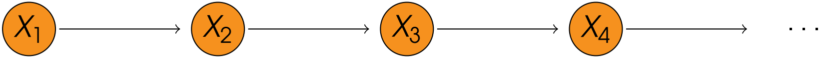
$$\mathbb{E}(X_t \mid \mathcal{F}_{t-1}) = \lambda_t = \underbrace{\nu_t}_{\text{"endemic"}} + \underbrace{\phi_t X_{t-1}}_{\text{"epidemic"}} .$$

$$\log(\nu_t) = \alpha_\nu + \gamma_\nu \sin(2\pi t/52) + \delta_\nu \cos(2\pi t/52)$$

$$\log(\phi_t) = \alpha_\nu + \gamma_\phi \sin(2\pi t/52) + \delta_\phi \cos(2\pi t/52)$$

Simple Markov structure:



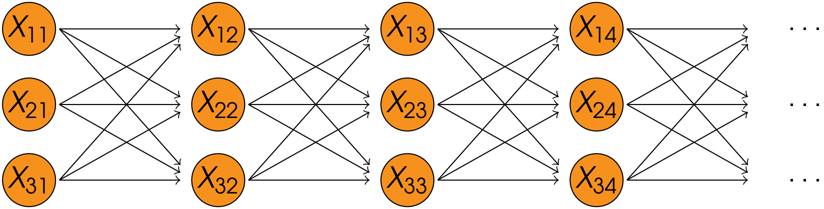
Moving to the multivariate case (Paul and Held 2008)

We model counts X_{it}, from regions $i = 1, \dots, m$ as

$$X_{it} \mid \mathcal{F}_{t-1} \sim \text{NegBin}(\lambda_{it}, \psi_i)$$

$$\lambda_{it} = \nu_{it} + \phi_{it} \sum_{j=1}^{m} w_{ji} X_{j,t-d}$$

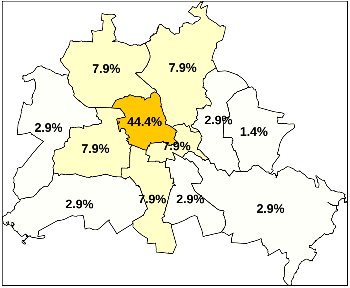
assuming $X_{it} \perp X_{jt} \mid \mathcal{F}_{t-1}$.

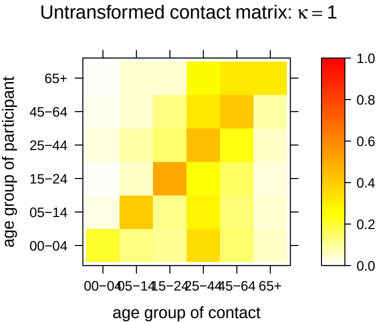


See Pedeli and Karlis (2014, conference presentation) for a similar INAR-type model

Examples for weight specification

- Spatial units: spatial power law based on path distances between regions
- Age groups: (transformed) social contact matrices





A bivariate example: norovirus in Bremen and Lower Saxony

$$\text{Bremen: } X_{Bt} \mid \mathcal{F}_{t-1} \sim \text{NBin}(\lambda_{Bt}, \psi_B),$$
$$\lambda_{Bt} = \nu_{Bt} + \phi_{BBt} X_{B,t-1} + \phi_{LB} X_{L,t-1}$$

$$\text{Lower Saxony: } X_{Lt} \mid \mathcal{F}_{t-1} \sim \text{NBin}(\lambda_{Lt}, \psi_L),$$
$$\lambda_{Lt} = \nu_{Lt} + \phi_{LLt} X_{L,t-1} + \phi_{BL} X_{B,t-1}$$

with periodically varying parameters

$$\log(\nu_{Bt}) = \log(e_B) + \alpha_B^{(\nu)} + \gamma^{(\nu)} \sin(\omega t) + \delta^{(\nu)} \cos(\omega t)$$
$$\log(\nu_{Lt}) = \log(e_L) + \alpha_L^{(\nu)} + \gamma^{(\nu)} \sin(\omega t) + \delta^{(\nu)} \cos(\omega t)$$

$\log(e_i)$: population offset

A bivariate example: norovirus in Bremen and Lower Saxony

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$$\lambda_{Lt} = \nu_{Lt} + \phi_{LLt} X_{L,t-1} + \phi_{BL} X_{B,t-1}$$

with periodically varying parameters

$$\log(\phi_{BBt}) = \alpha_B^{(\phi)} + \beta^{(\phi)} c_t + \gamma^{(\phi)} \sin(\omega t) + \delta^{(\phi)} \cos(\omega t)$$
$$\log(\phi_{LLt}) = \alpha_L^{(\phi)} + \beta^{(\phi)} c_t + \gamma^{(\phi)} \sin(\omega t) + \delta^{(\phi)} \cos(\omega t)$$

c_t: indicator for Christmas break (weeks 52 and 1)

A bivariate example: norovirus in Bremen and Lower Saxony

$$\text{Bremen: } X_{Bt} \mid \mathcal{F}_{t-1} \sim \text{NBin}(\lambda_{Bt}, \psi_B),$$
$$\lambda_{Bt} = \nu_{Bt} + \phi_{BBt} X_{B,t-1} + \phi_{LB} X_{L,t-1}$$

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$$\lambda_{Lt} = \nu_{Lt} + \phi_{LLt} X_{L,t-1} + \phi_{BL} X_{B,t-1}$$

with periodically varying parameters

$$\log(\phi_{LBt}) = \log(e_B) + \quad \alpha_\times^{(\phi)} + \gamma_\times^{(\phi)} \sin(\omega t) + \delta_\times^{(\phi)} \cos(\omega t)$$
$$\log(\phi_{BLt}) = \log(e_L) + \quad \alpha_\times^{(\phi)} + \gamma_\times^{(\phi)} \sin(\omega t) + \delta_\times^{(\phi)} \cos(\omega t)$$

$\log(e_i)$: population offset

Fitting models using the `surveillance` package

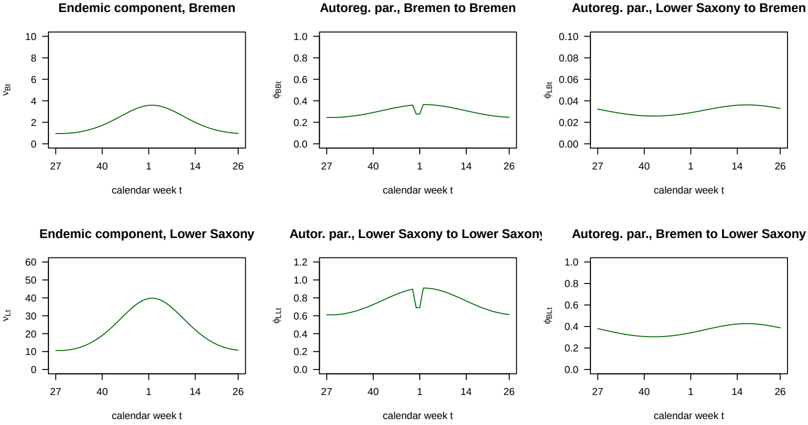
- Conditional likelihood inference is implemented in the `surveillance` package in `R`
- Also allows for random effects (via penalized maximum likelihood inference)
- Very fast as Hessian of log-likelihood is available analytically
- Other authors (Bauer and Wakefield 2018, Wakefield 2018) use full Bayesian inference with `Stan` instead

Implementation in `surveillance`

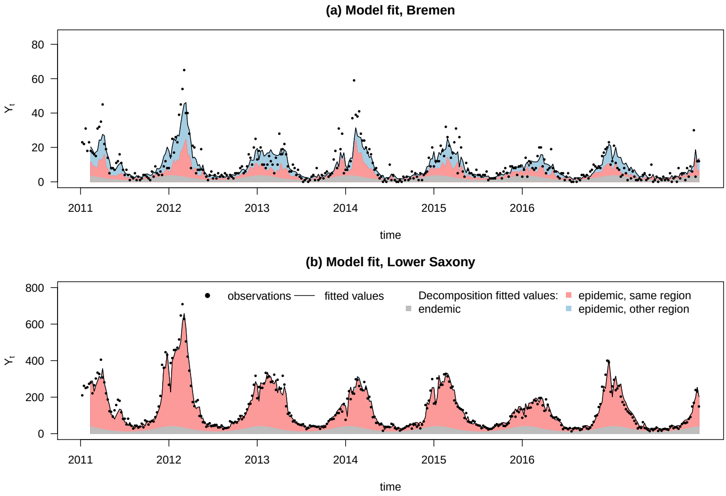
```
library(surveillance); library(hhh4addon); data("noroBL")
ctrl <- list(end = list(f = addSeason2formula(~ -1 + fe(1, unitSpecific = TRUE), S = 1)),
             ar = list(f = addSeason2formula(~ -1 + fe(1, unitSpecific = TRUE) +
                                                  fe(x, unitSpecific = FALSE), S = 1)),
             ne = list(f = addSeason2formula(~ 1, S = 1), offset = offsets_ne),
             subset = 6:nrow(noroBL@observed), family = "NegBinM")
fit <- hhh4(stsObj = noroBL, control = ctrl); summary(fit)

##
## Call:
## hhh4(stsObj = noroBL, control = ctrl)
##
## Coefficients:
##                          Estimate   Std. Error
## ar.sin(2 * pi * t/52)     0.027072   0.042768
## ar.cos(2 * pi * t/52)     0.197690   0.053617
## ar.1.Bremen              -1.207320   0.187138
## ar.1.Lower.Saxony        -0.294555   0.045122
## ar.x                     -0.271165   0.089355
## ne.1                     -7.692599   0.238927
## ne.sin(2 * pi * t/52)     0.160439   0.205806
## ne.cos(2 * pi * t/52)    -0.053408   0.115579
## end.sin(2 * pi * t/52)    0.074982   0.121339
## end.cos(2 * pi * t/52)    0.659331   0.159525
## end.1.Bremen              0.615242   0.290043
## end.1.Lower.Saxony        3.021059   0.132665
## overdisp.Bremen           0.157978   0.022137
## overdisp.Lower.Saxony     0.043781   0.004281
##
## Log-likelihood:   -2593.33
## AIC:              5214.66
## BIC:              5278.73
##
## Number of units:        2
## Number of time points:  359
```

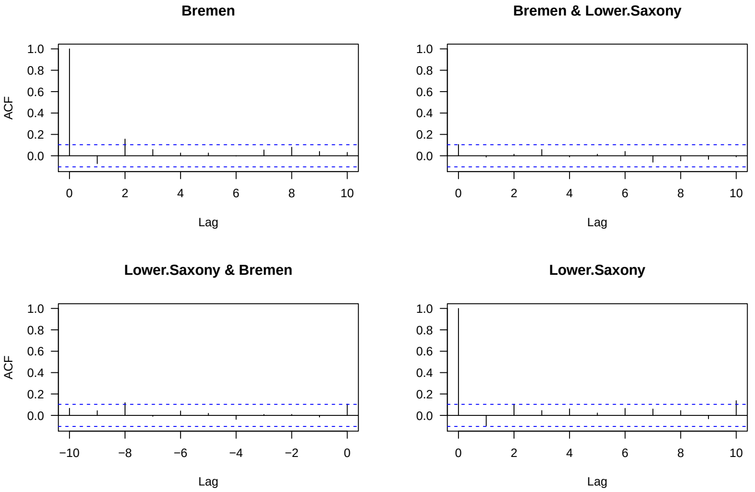
Estimated seasonally varying parameters



Display of model fit



Autocorrelation structure of the Pearson residuals



Adding higher-order lags to the model

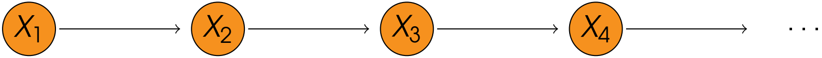
Feedback mechanism in the univariate model

We model the number X_t of new cases in week t as

$$X_t \mid \mathcal{F}_{t-1} \sim \text{NegBin}(\lambda_t, \psi)$$

$$\lambda_t = \nu_t + \phi_t X_{t-1}$$

Simple Markov structure:



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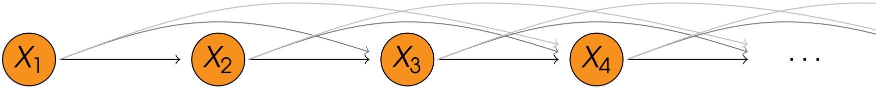
$$X_t \mid \mathcal{F}_{t-1} \sim \text{NegBin}(\lambda_t, \psi)$$

$$\lambda_t = \nu_t + \phi_t X_{t-1} + \kappa\lambda_{t-1}$$

→ negative binomial INGARCH(1, 1) model (Zhu 2011) with periodic structure

Higher-order dependencies:



Feedback mechanism in the univariate model

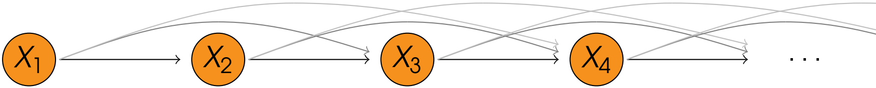
We model the number X_t of new cases in week t as

$$X_t \mid \mathcal{F}_{t-1} \sim \text{NegBin}(\lambda_t, \psi)$$

$$\lambda_t = \nu_t^* + \sum_{d=1}^{\infty} \phi_{t-d+1} \kappa^{d-1} X_{t-d}$$

→ negative binomial INGARCH(1, 1) model (Zhu 2011) with periodic structure

Higher-order dependencies:



Feedback mechanism in the univariate model

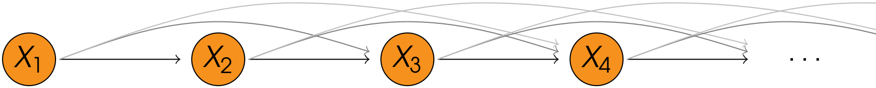
We model the number X_t of new cases in week t as

$$X_t \mid \mathcal{F}_{t-1} \sim \text{NegBin}(\lambda_t, \psi)$$

$$\lambda_t = \nu_t^* + \underbrace{\phi(1-\kappa)^{-1}}_{\phi^*} \sum_{d=1}^{\infty} \underbrace{(1-\kappa)\kappa^{d-1}}_{v_d} X_{t-d}$$

for simplified model with $\phi_t \equiv \phi$

Higher-order dependencies:

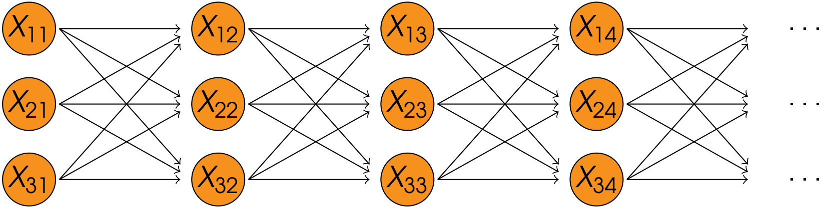


Weighted lags in the multivariate case

We model counts X_{it}, from regions $i = 1, \dots, m$ as

$$X_{it} \mid \mathcal{F}_{t-1} \sim \text{NegBin}(\lambda_{it}, \psi_i)$$

$$\lambda_{it} = \nu_{it} + \phi_{it} \sum_{j=1}^{m} w_{ji} X_{j,t-1}$$



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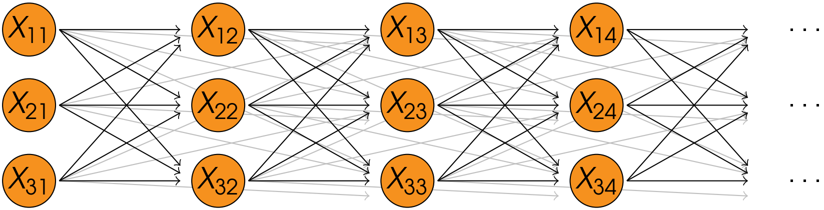
$$\lambda_{it} = \nu_{it} + \phi_{it} \sum_{j=1}^{m} w_{ji} \sum_{d=1}^{p} \lfloor v_d \rfloor X_{j,t-d}$$

with stand. non-neg. weights $0 \leqslant \lfloor v_d \rfloor = v_d / \sum_{d=1}^{p} v_d$



Parameterizing the weights v_d

For infectious disease counts the v_d, $d = 1, 2, \dots$ essentially reflect the serial interval distribution.

Possible specifications:

- geometric: $v_d = (1 - \alpha)\alpha^{d-1}$
- Poisson (Kucharski et al 2014): $v_d = \frac{\alpha^{d-1}}{(d-1)!}\exp(-\alpha)$
- multinomial (Forsberg White and Pagano 2008)
- $\dots$

A pragmatic and generic extension of the fitting algorithm

- package `hhh4addon` implemented on top of existing package `surveillance`:
  - For fixed α the conditional ML estimation procedure is easily adapted and still fast
  - Use profile likelihood approach to estimate α
  - To account also for uncertainty in α in estimated standard errors: obtain observed Fisher information (of all parameters, including α) numerically
- Advantage: allows user to specify weight-function (with parameter α)

Implementation in hhh4addon

```
fit2 <- profile_par_lag(stsObj = noroBL, control = ctrl); summary(fit2$best_mod)

##
## Call:
## hhh4_lag(stsObj = stsObj, control = control)
##
## Coefficients:
##                          Estimate   Std. Error
## ar.sin(2 * pi * t/52)   -0.066414   0.038601
## ar.cos(2 * pi * t/52)    0.242404   0.056945
## ar.1.Bremen             -0.828386   0.157620
## ar.1.Lower.Saxony       -0.197872   0.043073
## ar.x                    -0.371474   0.075186
## ne.1                    -7.840447   0.303766
## ne.sin(2 * pi * t/52)    0.071704   0.250370
## ne.cos(2 * pi * t/52)   -0.022062   0.134052
## end.sin(2 * pi * t/52)   0.063466   0.173581
## end.cos(2 * pi * t/52)   0.518644   0.266607
## end.1.Bremen             0.286563   0.453704
## end.1.Lower.Saxony       2.640002   0.199398
## overdisp.Bremen          0.150997   0.021484
## overdisp.Lower.Saxony    0.040270   0.003998
##
## Distributed lags used (max_lag = 5). Weights: 0.69; 0.22; 0.07; 0.02; 0.01
## Use distr_lag() to check the applied lag distribution and parameters.
##
## Log-likelihood:   -2577.58
## AIC:              5183.17
## BIC:              5247.24
##
## Number of units:        2
## Number of time points:  359
```

Implementation in hhh4addon

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fit2 <- profile_par_lag(stsObj = noroBL, control = ctrl); summary(fit2$best_mod)

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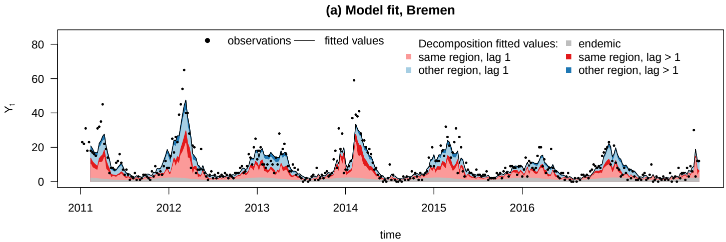
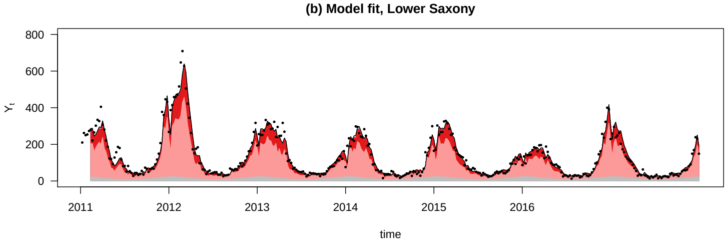
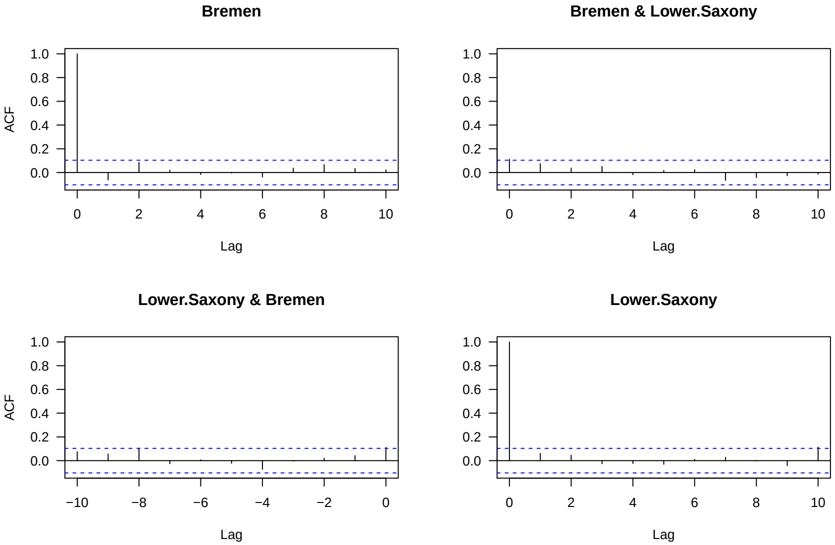
##
## Log-likelihood:   -2577.58
## AIC:              5183.17   ## AR(1) model: 5214.66
## BIC:              5247.24   ## AR(1) model: 5278.73
##
## Number of units:        2
## Number of time points:  359
```

Display of model fit with geometric lag





Autocorrelation of Pearson residuals



Periodic stationarity properties

Definitions: Periodic stationarity

- **periodic stationarity in the mean:** there exist $\mu_t = \mathbb{E}(\mathbf{X}_t)$ which are periodic (with period S), i.e. $\mu_t = \mu_{t+S}$
- **second-order periodic stationarity:** in addition there exist $\text{Cov}(\mathbf{X}_t, \mathbf{X}_{t'})$ which are periodic, i.e.

$$\text{Cov}(\mathbf{X}_t, \mathbf{X}_{t'}) = \text{Cov}(\mathbf{X}_{t+S}, \mathbf{X}_{t'+S})$$

Univariate model

Consider a NBin-INGARCH(1, 1) model $\{X_t; t \in \mathbb{Z}\}$ with

$$X_t \sim \text{NegBin}(\lambda_t, \psi_t)$$
$$\lambda_t = \nu_t + \phi_t X_{t-1} + \kappa_t \lambda_{t-1}$$

where the parameters $\nu_t, \phi_t, \kappa_t, \psi_t$ vary periodically with period S.

Notational convention in the following: set $\prod_{i=j}^{k} x_i = 1$ if $j > k$.

Compare Bentarzi and Bentarzi (2017) for the (univariate) Poisson case.

Periodic stationarity in the univariate model (I)

The process $\{X_t, t \in \mathbb{Z}\}$ is periodically stationary in the mean if $\prod_{m=1}^{S}(\phi_m + \kappa_m) < 1$. The unconditional means are

$$\mathbb{E}(X_t) = \mu_t = \frac{\sum_{i=0}^{S-1}\left\{\nu_{t-i}\prod_{m=0}^{i-1}(\phi_{t-m} + \kappa_{t-m})\right\}}{1 - \prod_{m=1}^{S}(\phi_m + \kappa_m)}$$

Periodic stationarity in the univariate model (II)

The process $\{X_t, t \in \mathbb{Z}\}$ is periodically stationary of second order if $\prod_{m=1}^{S} k_m < 1$ with $k_m = \left\{(\phi_m + \kappa_m)^2 + \phi_m^2 \psi_{m-1}\right\}$.

The unconditional variances are

$$\begin{aligned}
\text{Var}(X_t) = \sigma_t^2 &= (\mu_t + \psi_t \mu_t^2) + (1 + \psi_t)\text{Var}(\lambda_t) \\
&= (\mu_t + \psi_t \mu_t^2) + \\
&\quad (1 + \psi_t) \, \frac{\sum_{i=0}^{S-1} \left\{\phi_{t-i}^2 (\mu_{t-i-1} + \psi_{t-i-1} \mu_{t-i-1}^2) \prod_{m=0}^{i-1} k_{t-m}\right\}}{1 - \prod_{m=1}^{S} k_m}
\end{aligned}$$

Periodic stationarity in the univariate model (III)

The unconditional d-th order autocovariances, $d = 1, 2, \dots$ are

$$\text{Cov}(X_t, X_{t-d}) =$$

$$\left\{\phi_{t-d+1}\sigma^2_{t-d} + \kappa_{t-d+1}\frac{\sigma^2_{t-d} - (\mu_{t-d} + \psi_{t-d}\mu^2_{t-d})}{\psi_{t-d} + 1}\right\}\prod_{i=0}^{d-2}(\phi_{t-i} + \kappa_{t-i})$$

The multivariate case (I)

Consider a multivariate NBin-INARCH(p) model $\{\mathbf{X}_t, t \in \mathbb{Z}\}$:

$$X_{it} \mid \mathbf{X}_{t-1} \sim \text{NegBin}(\lambda_{it}, \psi_{it})$$
$$\boldsymbol{\lambda}_t = \boldsymbol{\nu}_t + \boldsymbol{\phi}_{1t}\mathbf{X}_{t-1} + \ldots + \boldsymbol{\phi}_{pt}\mathbf{X}_{t-p}$$

$$X_{it} \perp X_{jt} \mid \mathcal{F}_{t-1}$$

where $\boldsymbol{\nu}_t$, $\boldsymbol{\phi}_{dt}$, $d = 1, \ldots, p$, and $\boldsymbol{\psi}_t$ vary periodically with period S.

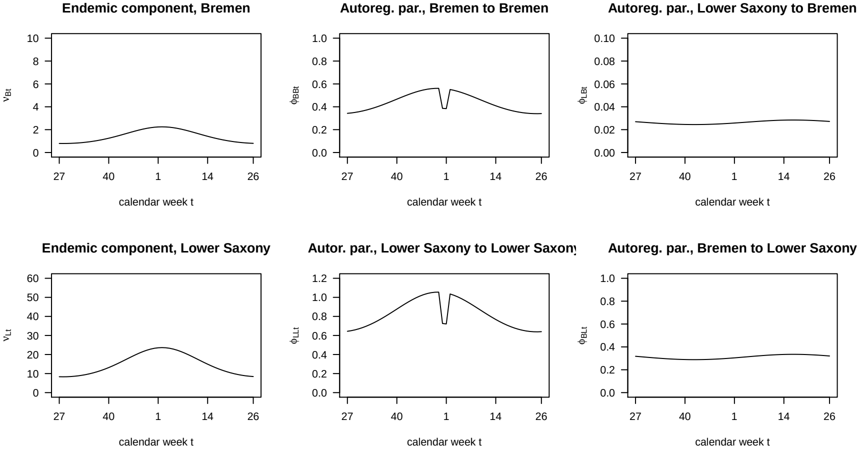
Obtaining periodically stationary moments in the multivariate case, $p \geqslant 1$

- A criterion for periodic stationarity in the mean follows directly from Ula and Smadi (1997; based on stationarity of a lumped PAR(1) representation)
- Periodically stationary means can be obtained by solving an eigenvector problem
- Periodically stationary autocovariance functions can be obtained numerically using a recursive relationship

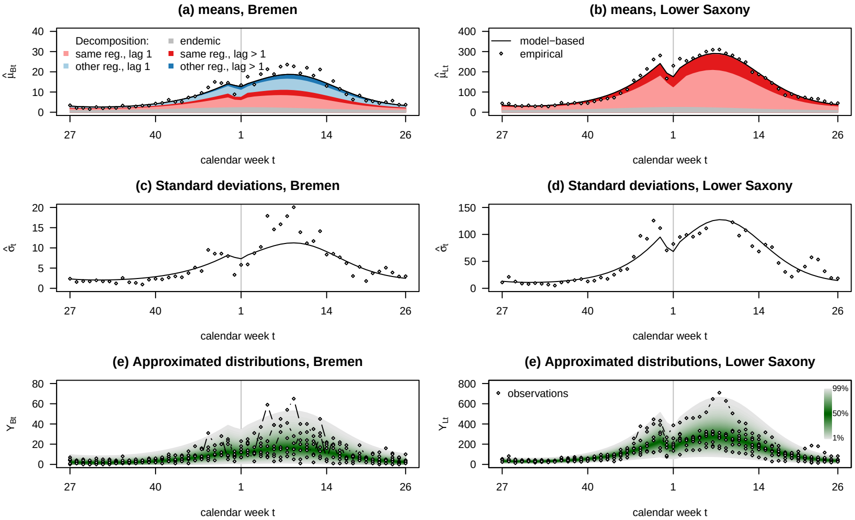
What is the use of knowing periodically stationary moments?

- model description and comparison
- (predictive model assessment)
- outbreak detection?

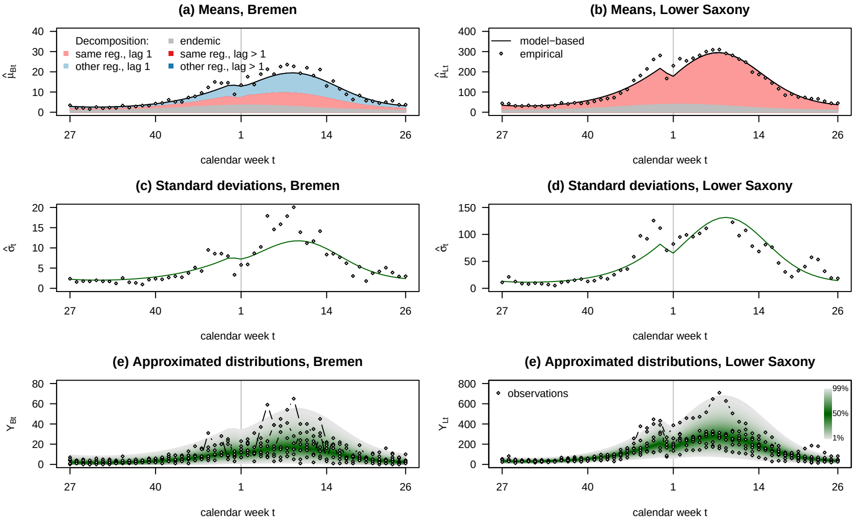
Estimated parameters in the AR(p) model



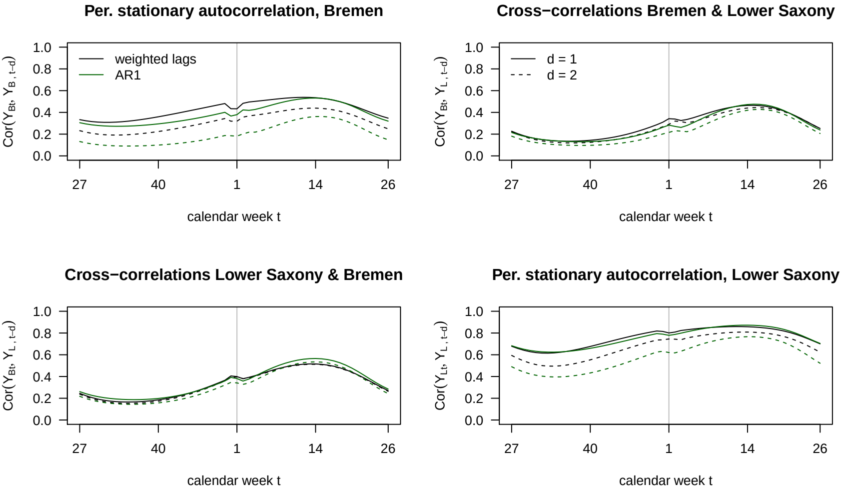
Periodically stationary moments of the AR(p) lags model



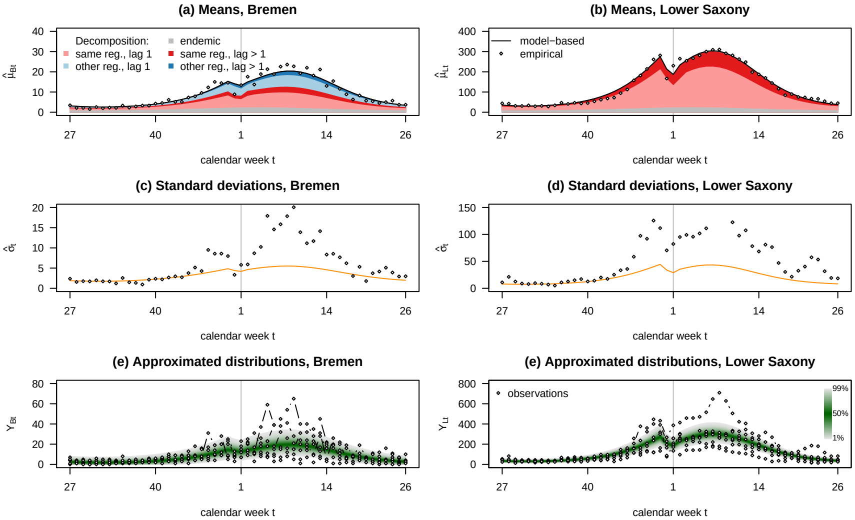
Periodically stationary moments of the AR-1 model



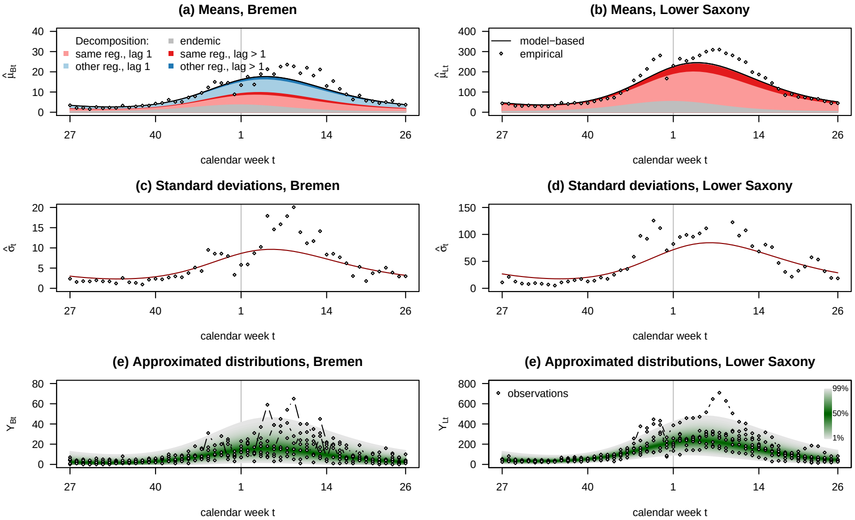
Periodically stationary autocorrelation structure



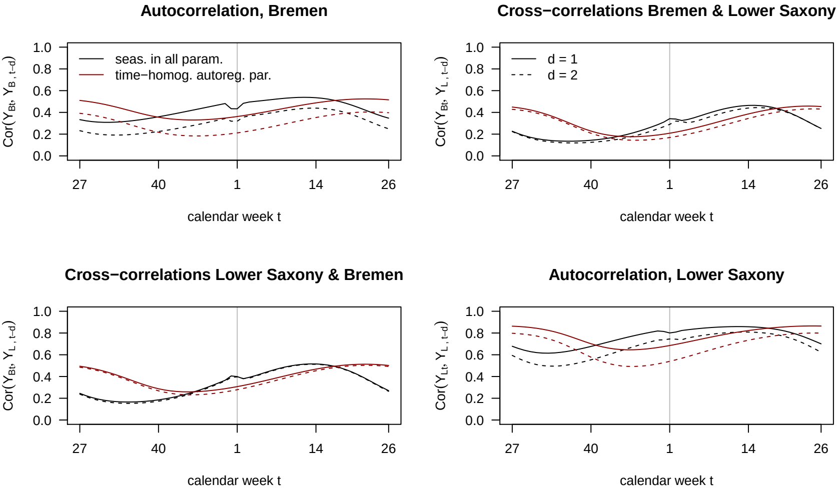
Periodically stationary moments of a Poisson version



Periodically stationary moments of a model with time-constant autoregressive parameters



Periodically stationary autocorrelation structure



An idea for outbreak detection

Common strategy to detect outbreak in week t^*:

- Fit a model to past data (e.g. up to $t^* - 1$ or $t^* - 26$)
- Obtain a reference/forecast distribution for incidence in week t^*
- If the observed X_{t^*} exceeds a certain quantile of this distribution: flag an outbreak

Strategy 1: seasonal forecast

The Farrington algorithm

1. Fit a quasi-Poisson model

$$\log(\mu_t) = \alpha + \beta t + \delta_{m(t)}$$

   with a time trend and zero-order splines for seasonality to the data up to week $t^* - 26$.
2. Re-fit the model downweighting past outbreaks (based on Anscombe residuals).
3. Construct one-sided $(1 - \alpha) \cdot 100\%$ prediction interval $[0, U_{t^*}]$ for X_{t^*}.
4. Flag an alarm if $X_{t^*} > U_{t^*}$.

$\rightarrow$ detects excess relative to seasonal average

Strategy 2: one-step-ahead forecast

E.g. Pedeli and Karlis (2014)

1. Fit an INAR-type model to past data up to $t^* - 1$
2. Construct one-sided one-step-ahead $(1 - \alpha) \cdot 100\%$ prediction interval $[0, U_{t^*}]$ for X_{t^*}.
3. Flag an alarm if $X_{t^*} > U_{t^*}$.

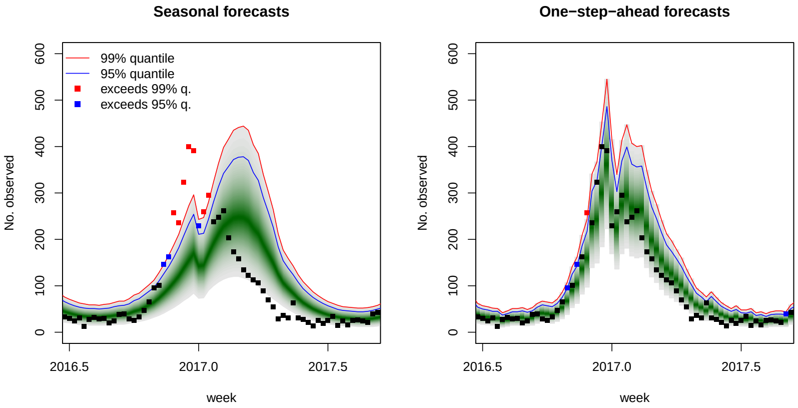
$\rightarrow$ detects excess relative to immediate past

Outbreak detection based on endemic-epidemic model

Our results allow to generate two different reference distributions from the same model:

- **one-step-ahead** forecast to detect excess relative to immediate past
- (approximated) **periodically stationary** distribution to detect excess relative to seasonal average

Example: early onset of norovirus in autumn 2016



Many open questions...

- Remove past outbreaks from training data?
- Which quantile should be used as threshold?
- Multiple testing?
- ...

Conclusion

- **Higher-order lags** can considerably improve model fits (and also prediction)
- **Periodically stationary moments** are easy to obtain numerically and are useful for model interpretation
- Moreover they open an interesting pathway to **outbreak detection**.

References

- Bauer, C. and Wakefield, J. (2018). Stratified spacetime infectious disease modelling, with an application to hand, foot and mouth disease in china. Journal of the Royal Statistical Society: Series C (Applied Statistics), (available online first, DOI 10.1111/rssc.12284).
- Bentarzi, M. and Bentarzi, W. (2017). *Periodic integer-valued GARCH(1, 1) model*. Communications in Statistics: Simululation and Computation, 46(2):1167–1188.
- Forsberg White, L. and Pagano, M. (2008). *A likelihood-based method for real-time estimation of the serial interval and reproductive number of an epidemic.* Statistics in Medicine, 27(16):2999–3016.

References (II)

- Held, L., Höhle, M., and Hofmann, M. (2005). *A statistical framework for the analysis of multivariate infectious disease surveillance counts.* Statistical Modelling, 5(3):187–199.
- Kucharski, A., Mills, H., Pinsent, A., Fraser, C., Van Kerkhove, M., Donnelly, C., and S., R. (2014). *Distinguishing between reservoir exposure and human-to-human transmission for emerging pathogens using case onset data.* PLOS Currents Outbreaks, 1.
- Meyer, S. and Held, L. (2014). *Power-law models for infectious disease spread.* Annals of Applied Statistics, 8(3):1612–1639.

References (III)

- Meyer, S., Held, L., and Höhle, M. (2017). *Spatio-temporal analysis of epidemic phenomena using the R package surveillance.* Journal of Statistical Software, 77(11):1–55.
- Meyer, S. and Held, L. (2017). *Incorporating social contact data in spatio-temporal models for infectious disease spread.* Biostatistics, 18(2):338–351.
- Paul, M. and Held, L. (2008) *Multivariate modelling of infectious disease surveillance data.* Statistics in Medicine 27:6250–6267.
- Pedeli, X., and Karlis, D. (2014). *Integer-valued autoregressive models: reliable and practical multivariate perspectives.* Statistische Woche 2014 16–19 September, Hannover

References (IV)

- Rao, Y., and McCabe, B. (2016) *Real-time surveillance for abnormal events: the case of influenza outbreaks.* Statistics in Medicine, 35: 2206–2220.
- Zhu, F. (2011). *A negative binomial integer-valued GARCH model.* Journal of Time Series Analysis, 32(1):54–67.

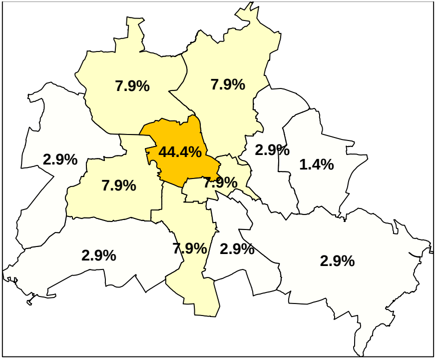
Appendix

Power-law parameterization for spatial models (Meyer and Held 2014)

For spatial units $i = 1, \ldots, m$:

$$w_{ij} = \frac{o_{ji}^{-\alpha}}{\sum_{k=1}^{m} o_{jk}^{-\alpha}}$$

where o_{ji} is the path distance between units j and i and α is estimated from the data.



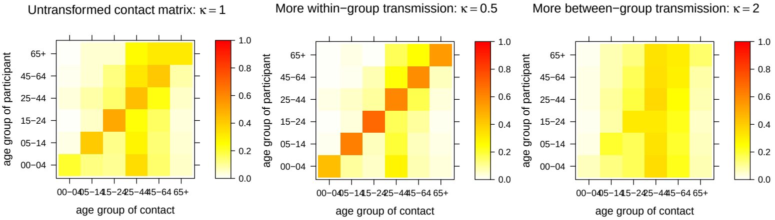
Social contact matrices for age-stratified models (Meyer and Held 2017)

For age groups $a = 1, \dots, m$:

$$w_{ab} = \frac{c_{ab}}{\sum_{k=1}^{m} c_{ak}}$$

where **C** is a social contact matrix.

C can be power-transformed to more within-group or more between-group transmission:



Per. stat. criterion from Ula and Smadi (1997)

Our model has a multivariate PAR(p) representation as

$$\mathbf{Y}_t = \boldsymbol{\nu}_t + \sum_{d=1}^{p} \phi_{t,d} \mathbf{Y}_{t-d} \quad + \mathbf{e}_t.$$

with uncorrelated white noise vectors $\mathbf{e}_t$; finiteness of $\mathrm{Cov}(\mathbf{e}_t)$ is not required for stationarity in the mean. Ula and Smadi (1997) give a necessary and sufficient condition for periodic stationarity in the mean. Consider the p-span lumped process

$$\mathbf{Z}_T = (\mathbf{Y}_{Tp}^\top, \mathbf{Y}_{Tp-1}^\top, \dots, \mathbf{Y}_{Tp-(p-q)}^\top)^\top$$
$$\mathbf{e}_T = (\mathbf{e}_{Tp}^\top, \mathbf{e}_{Tp-1}^\top, \dots, \mathbf{e}_{Tp-(p-q)}^\top)^\top$$
$$\boldsymbol{\nu}_T = (\boldsymbol{\nu}_{Tp}^\top, \boldsymbol{\nu}_{Tp-1}^\top, \dots, \boldsymbol{\nu}_{Tp-(p-q)}^\top)^\top$$

The mp-dimensional process $\{\mathbf{Z}_T\}$ then has a PAR(1) form with period s where s is chosen so that sp is the least common multiple of p and S. Specifically one can write

$$\mathbf{Z}_T = \boldsymbol{\nu}_T + \mathbf{A}_{T,0}\mathbf{A}_{T,1}\mathbf{Z}_{T-1} + \boldsymbol{\epsilon}_T$$

$$\mathbf{A}_{T,0} = \begin{pmatrix} \boldsymbol{I}_m & -\boldsymbol{\phi}_{Tp,1} & -\boldsymbol{\phi}_{Tp,2} & \cdots & -\boldsymbol{\phi}_{Tp,p-1} \\ \mathbf{0}_m & \boldsymbol{I}_m & -\boldsymbol{\phi}_{Tp-1,1} & \cdots & -\boldsymbol{\phi}_{Tp-1,p-2} \\ & & & \vdots & \\ \mathbf{0}_m & \mathbf{0}_m & \mathbf{0}_m & \cdots & \boldsymbol{I}_m \end{pmatrix}$$

$$\mathbf{A}_{T,1} = \begin{pmatrix} \boldsymbol{\phi}_{Tp,p} & \mathbf{0}_m & \mathbf{0}_m & \cdots & \mathbf{0}_m \\ \boldsymbol{\phi}_{Tp-1,p-1} & \boldsymbol{\phi}_{Tp-1,p} & \mathbf{0}_m & \cdots & \mathbf{0}_m \\ & & & \vdots & \\ \boldsymbol{\phi}_{Tp-(p-1),1} & \boldsymbol{\phi}_{Tp-(p-1),2} & \boldsymbol{\phi}_{Tp-(p-1),3} & \cdots & \boldsymbol{\phi}_{Tp-(p-1),p} \end{pmatrix}.$$

The necessary and sufficient condition for the existence of the limit $\lim_{k\to\infty} \mathrm{E}(\mathbf{Z}_{ks}) = \boldsymbol{\mu}_s^Z$ is that all eigenvalues of $\mathbf{B} = \mathbf{A}_{s,0}^{-1}\mathbf{A}_{s,1} \cdots \mathbf{A}_{2,0}^{-1}\mathbf{A}_{2,1}\mathbf{A}_{1,0}^{-1}\mathbf{A}_{1,1}$ are smaller than 1 in absolute values; the existence of $\boldsymbol{\mu}_1^Z, \ldots, \boldsymbol{\mu}_{s-1}^Z$ follows directly.

Obtaining periodically stationary moments in the multivariate case, $p \geqslant 1$

Some notation:

$$\tilde{\mathbf{X}}_t = (1, \mathbf{X}_{t-p+1}^\top, \ldots, \mathbf{X}_t^\top)^\top$$
$$\tilde{\boldsymbol{\mu}}_t = \mathbb{E}(\tilde{\mathbf{X}}_t)$$
$$\tilde{\boldsymbol{\phi}}_t = \begin{pmatrix} 1 & \mathbf{0}_{1\times G} & \mathbf{0}_{1\times(p-1)G} & & \\ \mathbf{0}_{(p-1)G\times 1} & \mathbf{0}_{(p-1)G\times G} & \mathbf{I}_{(p-1)G\times(p-1)G} & & \\ \boldsymbol{\nu}_t & \boldsymbol{\phi}_{pt} & \boldsymbol{\phi}_{p-1,t} & \cdots & \boldsymbol{\phi}_{1t} \end{pmatrix}$$
$$\tilde{\boldsymbol{\psi}}_t = (\mathbf{0}_{1+(G-1)p}^\top, \boldsymbol{\psi}_t^\top)^\top$$

These are constructed so that

$$\begin{aligned} \tilde{\boldsymbol{\phi}}_t \tilde{\mathbf{X}}_{t-1} &= \mathbb{E}(\tilde{\mathbf{X}}_t \,|\, \tilde{\mathbf{X}}_{t-1}) \\ &= (1, \mathbf{X}_{t-p+1}^\top, \ldots, \mathbf{X}_{t-1}^\top, \boldsymbol{\lambda}_t^\top)^\top \\ \tilde{\boldsymbol{\phi}}_t \tilde{\boldsymbol{\mu}}_{t-1} &= \tilde{\boldsymbol{\mu}}_t \end{aligned}$$

Periodically stationary means

A criterion for periodic stationarity in the mean is given in Ula and Smadi (1997; based on stationarity of a lumped PAR(1) representation).

Assuming that $\tilde{\mu}_1, \ldots, \tilde{\mu}_S$ exist we get

$$\begin{aligned}
\tilde{\mu}_S &= \tilde{\Phi}_S \tilde{\mu}_{S-1} \\
\tilde{\mu}_S &= \tilde{\Phi}_S \tilde{\Phi}_{S-1} \tilde{\mu}_{S-2} \\
&\vdots \\
\tilde{\mu}_S &= \tilde{\Phi}_S \tilde{\Phi}_{S-1} \cdots \tilde{\Phi}_1 \tilde{\mu}_S
\end{aligned}$$

i.e. an eigenvector problem.

Periodically stationary covariances

Again **assuming** second-order stationarity the following recursion holds for $\tilde{\boldsymbol{\Lambda}}_t = \mathrm{E}(\tilde{\mathbf{X}}_t \tilde{\mathbf{X}}_t^\top)$:

$$(\tilde{\Lambda}_t)_{ij} = \begin{cases} \{(\tilde{\psi}_t)_i + 1\}(\ddot{\Lambda}_t)_{ii} + (\ddot{\Lambda}_t)_{1i} & \text{if } i = j > m(p-1)+1 \\ (\ddot{\Lambda}_t)_{ij} & \text{otherwise} \end{cases} \tag{1}$$

where $\ddot{\boldsymbol{\Lambda}}_t = \tilde{\boldsymbol{\phi}}_t \tilde{\boldsymbol{\Lambda}}_{t-1} \tilde{\boldsymbol{\phi}}_t^\top$.

Applying this recursion iteratively until convergence allows us to compute $\tilde{\boldsymbol{\Lambda}}_1, \ldots, \tilde{\boldsymbol{\Lambda}}_S$ numerically.

Periodically stationary covariances

Again **assuming** second-order stationarity the following recursion holds for $\tilde{\boldsymbol{\Lambda}}_t = \mathsf{E}(\tilde{\mathbf{X}}_t \tilde{\mathbf{X}}_t^\top)$:

$$\ddot{\boldsymbol{\Lambda}}_t = \tilde{\boldsymbol{\phi}}_t \tilde{\boldsymbol{\Lambda}}_{t-1} \tilde{\boldsymbol{\phi}}_t^\top \tag{2}$$

$$\tilde{\boldsymbol{\Lambda}}_t = (\mathbf{1} + \mathrm{diag}(\tilde{\boldsymbol{\psi}}_t)) \circ \ddot{\boldsymbol{\Lambda}}_t + \mathrm{diag}(0, \tilde{\boldsymbol{\mu}}_t) \tag{3}$$

where $\ddot{\boldsymbol{\Lambda}}_t = \tilde{\boldsymbol{\phi}}_t \tilde{\boldsymbol{\Lambda}}_{t-1} \tilde{\boldsymbol{\phi}}_t^\top$.

Applying this recursion iteratively until convergence allows us to compute $\tilde{\boldsymbol{\Lambda}}_1, \ldots, \tilde{\boldsymbol{\Lambda}}_S$ numerically.