Lipschitz p-convex and q-concave maps

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Theorem (Ribe)

If two Banach spaces are uniformly homeomorphic, they are crudely finitely representable in each other.

Ribe's program (as described by Mendel and Naor)

"The search for purely metric reformulations of basic linear concepts and invariants from the local theory of Banach spaces".

Ribe's program

Some examples of successes:

1. Superreflexivity (Bourgain, 1986).
2. Equivalent norm with modulus of convexity of power type p (Lee/Naor/Peres, 2009; Mendel/Naor, 2008).
3. Rademacher cotype (Mendel/Naor, 2008).
4. Rademacher type (Mendel/Naor, 2007).

Ribe's program for maps

The local theory of Banach spaces is not only concerned with the spaces but also with the morphisms between them, and moreover there is a rich interplay between the properties of the spaces and those of the morphisms.

Nonlinear characterizations of linear properties for maps:

1. p-summing maps (Farmer/Johnson, 2009).
2. Factorization through a subspace of L_p (Johnson/Maurey/Schechtman, 2009).
3. p-nuclear operators (Chen/Zheng, 2012).
4. Modulus of convexity of power type p up to a renorming of the domain (C).
5. Rademacher cotype (C).
6. Rademacher type (C).

Banach lattice
=
Banach space + partial order compatible with the norm and the algebraic structure
(think of a Banach space of functions or sequences, like $L_p[0,1]$ or c_0).

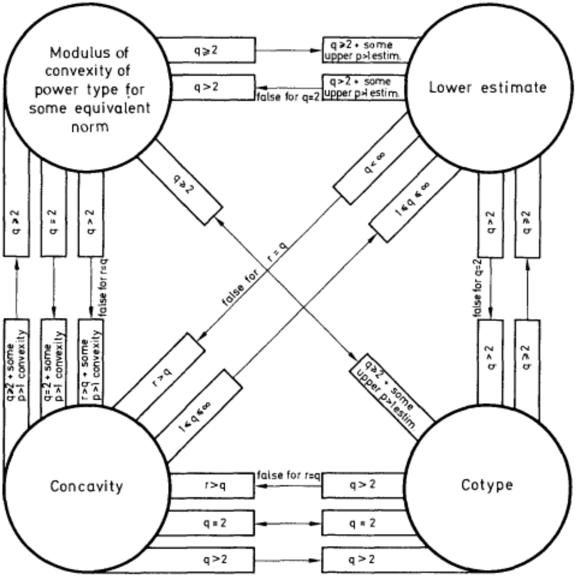
The important conditions:

- Both addition and multiplication by positive scalars preserve the order.
- For every x and y there exist a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.
- $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, where $|x| = x \vee (-x)$.

The beauty of Banach lattices

- Paraphrasing Lindenstrauss and Tzafriri: this additional ingredient makes the theory of Banach lattices in some regards simpler, cleaner and more complete than the theory for general Banach spaces.
- For Banach lattices, Rademacher type, Rademacher cotype and having an equivalent uniformly p-convex norm are intimately related to the notions of convexity and concavity.

Connections to other properties



The Krivine functional calculus for lattices

In a space of functions, say $C(K)$ for concreteness, we can consider expressions like

$$\Big(\sum_{j=1}^{n} |f_j|^p\Big)^{1/p}, \qquad f_1, \dots, f_n \in C(K).$$

Krivine developed a functional calculus that allows us to make sense of this expression in any Banach lattice.

Linear p-convex maps

Consider $1 \leq p \leq \infty$. A linear map $T : X \to E$ from a Banach space X to a Banach lattice E is called *p-convex* if there exists a constant $M < \infty$ such that for all $v_1, \ldots, v_n \in X$

$$\left\| \Big(\sum_{j=1}^{n} |Tv_j|^p \Big)^{1/p} \right\|_E \leq M \Big(\sum_{j=1}^{n} \|v_j\|_X^p \Big)^{1/p}, \quad \text{if } 1 \leq p < \infty$$

or

$$\left\| \bigvee_{j=1}^{n} |Tv_j| \right\|_E \leq M \max_{1 \leq j \leq n} \|v_j\|_X , \quad \text{if } p = \infty.$$

The smallest such constant M is denoted $M^{(p)}(T)$.

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- IDEA: T induces a bounded map $\ell_p(X) \to E(\ell_p)$.

Linear q-concave maps

A linear map $S : E \to Y$ from a Banach lattice E to a Banach space Y is called q-concave if there exists a constant $M < \infty$ such that for all $v_1, \dots, v_n \in E$,

$$\Big(\sum_{j=1}^{n} \|Sv_j\|_Y^q \Big)^{1/q} \leq M \left\| \Big(\sum_{j=1}^{n} |v_j|^q \Big)^{1/q} \right\|_E , \qquad \text{if } 1 \leq q < \infty$$

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$$\max_{1 \leq j \leq n} \|Sv_j\|_Y \leq \left\| \bigvee_{j=1}^{n} |v_j| \right\|_E , \qquad \text{if } q = \infty.$$

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The smallest such constant M is denoted $M_{(q)}(S)$.

- IDEA: S induces a bounded map $E(\ell_q) \to \ell_q(Y)$.

A factorization theorem (Nikishin/Maurey)

Let E be a Banach space, (Ω, Σ, μ) a σ-finite measure space, $1 \leq p < q < \infty$, $T : E \to L_p(\mu)$ a linear map and $0 < C < \infty$. TFAE:

(a) There exists a density function h on Ω and a linear map $S : E \to L_q(hd\mu)$ with $\|S\| \leq C$ such that

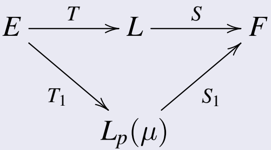
$$\begin{array}{ccc} E & \xrightarrow{\quad T \quad} & L_p(\mu) \\ \Big\downarrow S & & \Big\uparrow j \\ L_q(hd\mu) & \xrightarrow{\quad i_{q,p} \quad} & L_p(hd\mu) \end{array}$$

(b) For every $x_1, \ldots, x_n$ in E, $\left\| \left(\sum_{j=1}^{n} |Tx_j|^q \right)^{1/q} \right\|_{L_p(\mu)} \leq C \left(\sum_{j=1}^{n} \|x_j\|^q \right)^{1/q}.$

Factorization through L_p

Theorem (Krivine)

Let E, F be Banach spaces and L a Banach lattice. Suppose that $T : E \to L$ is p-convex and $S : L \to F$ is p-concave. Then the operator ST can be factored through an $L_p(\mu)$ space. Moreover, we may arrange to have $ST = S_1 T_1$ with $T_1 : E \to L_p(\mu)$, $S_1 : L_p(\mu) \to F$, $\|T_1\| \le M^{(p)}(T)$ and $\|S_1\| \le M_{(p)}(S)$.



Goals

- I did not try yet to obtain a fully nonlinear characterization of convexity/concavity.
- I started by working on a partially nonlinear situation, with maps between metric spaces and Banach lattices.
- I was aiming for nonlinear factorization theorems.

Let $1 \le p \le \infty$. Let X be a metric space and E a Banach lattice. A Lipschitz map $T : X \to E$ is called *Lipschitz p-convex* if there exists a constant $C \ge 0$ for any $x_j, x_j' \in X$ and $\lambda_j \ge 0$,

$$\left\| \left(\sum_{j=1}^{n} \lambda_j |Tx_j - Tx_j'|^p \right)^{1/p} \right\|_E \le C \left(\sum_{j=1}^{n} \lambda_j d(x_j, x_j')^p \right)^{1/p},$$

(with the obvious adjustment if $p = \infty$). The smallest such constant C is called the *Lipschitz p-convexity constant* of T and is denoted by $M_{\mathrm{Lip}}^{(p)}(T)$.

Remark

It would be nice to apply the usual Farmer/Johnson/Mendel/Schechtman argument and obtain that in the definition of Lipschitz p-convexity it suffices to consider $\lambda_j = 1$, i.e.

$$\left\| \left(\sum_{j=1}^{n} |Tx_j - Tx_j'|^p \right)^{1/p} \right\|_E \leq C \left(\sum_{j=1}^{n} d(x_j, x_j')^p \right)^{1/p}.$$

That is indeed the case if the lattice has certain continuity properties, but let's avoid the technicalities.

Nonlinear Maurey/Nikishin factorization (C)

Let X be a metric space, (Ω, Σ, μ) a σ-finite measure space, $1 \leq p < q < \infty$, $T : X \to L_p(\mu)$ a Lipschitz map and $0 < C < \infty$. TFAE:

(a) There exists a density function h on Ω and a Lipschitz map $S : X \to L_q(hd\mu)$ with $\mathrm{Lip}(S) \leq C$ such that

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(b) For every $x_j, x_j' \in X$ and $\lambda_j \geq 0$,

$$\left\| \left(\sum_{j=1}^{n} \lambda_j |Tx_j - Tx_j'|^q \right)^{1/q} \right\|_{L_p(\mu)} \leq C \left(\sum_{j=1}^{n} \lambda_j d(x_j, x_j')^q \right)^{1/q}.$$

Molecules and the Lipschitz-free space

- A molecule on a metric space X is $m : X \to \mathbb{R}$ such that

$$\sum_{x \in X} m(x) = 0.$$

- Those of the form

$$m_{xx'} := \chi_{\{x\}} - \chi_{\{x'\}}$$

  with $x, x' \in X$ are called *atoms*.
- The Lipschitz-free space of X is (the completion of) the space of molecules with the norm

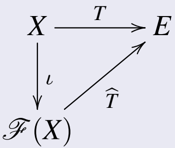
$$\|m\|_{\mathscr{F}(X)} := \inf\left\{ \sum_{j=1}^{n} |a_j| d(x_j, x_j') \ : \ m = \sum_{j=1}^{n} a_j m_{x_j x_j'} \right\}.$$

- $\mathscr{F}(X)^* = \mathrm{Lip}_0(X)$.

Universal property of Lipschitz-free spaces

Theorem (Arens/Eells)

Let X be a metric space with a designated point $0 \in X$. The map $\iota : x \mapsto m_{x0}$ is an isometric embedding of X into $\mathscr{F}(X)$. Moreover, for any Banach space E and any Lipschitz map $T : X \to E$ with $T(0) = 0$ there is a unique linear map $\widehat{T} : \mathscr{F}(X) \to E$ such that $\widehat{T} \circ \iota = T$. Furthermore, $\|\widehat{T}\| = \mathrm{Lip}(T)$.



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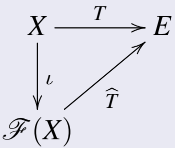
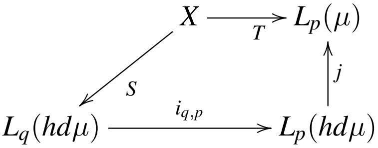


NOTE: To evaluate the norm of the linear extension $\widehat{T}$, it suffices to look at the images of atoms.

Relating the linear and nonlinear theorems

Let X be a metric space, (Ω, Σ, μ) a σ-finite measure space, $1 \leq p < q < \infty$, $T : X \to L_p(\mu)$ a Lipschitz map.



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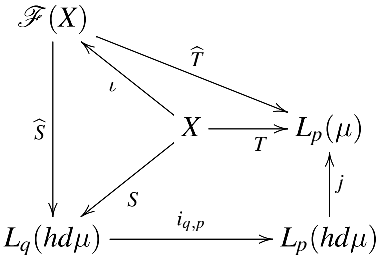

A consequence of the linear theorem

Let X be a metric space, (Ω, Σ, μ) a σ-finite measure space, $1 \le p < q < \infty$, $T : X \to L_p(\mu)$ a Lipschitz map and $0 < C < \infty$. TFAE:

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(b) For every $m_1, \dots, m_n \in \mathscr{F}(X)$,

$$\left\| \left(\sum_{j=1}^n |\hat{T} m_j|^q \right)^{1/q} \right\|_{L_p(\mu)} \le C \left(\sum_{j=1}^n \|m_j\|_{\mathscr{F}(X)}^q \right)^{1/q},$$ where

$\hat{T} : \mathscr{F}(X) \to L_p(\mu)$ is the linearization of T.

Putting everything together

Corollary (C)

Let X be a metric space, (Ω, Σ, μ) a σ-finite measure space, $1 \leq p < q < \infty$, $T : X \to L_p(\mu)$ a Lipschitz map and $0 < C < \infty$. TFAE:

(a) *For every $m_1, \dots, m_n \in \mathscr{F}(X)$,*

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where $\hat{T} : \mathscr{F}(X) \to L_p(\mu)$ is the linearization of T.

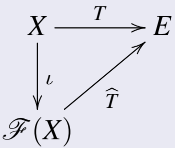
(b) *For every $x_j, x_j' \in X$ and $\lambda_j \geq 0$,*

$$\left\| \left(\sum_{j=1}^{n} \lambda_j |T x_j - T x_j'|^q \right)^{1/q} \right\|_{L_p(\mu)} \leq C \left(\sum_{j=1}^{n} \lambda_j d(x_j, x_j')^q \right)^{1/q}.$$

General equivalence

Theorem (C)

Let X be a metric space and E a Banach lattice. A Lipschitz map $T : X \to E$ is Lipschitz p-convex if and only if $\hat{T} : \mathscr{F}(X) \to E$ is p-convex. Moreover, in this case the p-convexity constants are the same.



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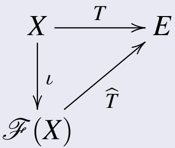

The “if” part is trivial: p-convexity of $\hat{T}$ clearly implies Lipschitz p-convexity of T with no increment in the constant, since $\|m_{xx'}\|_{\mathscr{F}(X)} = d(x,x')$ and $\hat{T}m_{xx'} = Tx - Tx'$.

The proof

Now suppose that T is Lipschitz p-convex. The strategy of the proof will be to show that $\widehat{T}^* : E^* \to \mathscr{F}(X)^* = \mathrm{Lip}_0(X)$ is p'-concave.
Let $\varphi_j^* \in E^*$ be arbitrary. For any $x_j, x_j' \in X$ with $x_j \neq x_j'$ we have

$$\Big(\sum_j \Big| \frac{\langle \varphi_j^*, Tx_j - Tx_j' \rangle}{d(x_j, x_j')} \Big|^{p'} \Big)^{1/p'} = \sup_{\sum_j |\alpha_j|^p \leq 1} \sum_j \alpha_j \frac{\langle \varphi_j^*, Tx_j - Tx_j' \rangle}{d(x_j, x_j')}.$$

Using Hölder's inequality for lattices, the latter is bounded by

$$\sup_{\sum_j |\alpha_j|^p \leq 1} \Bigg(\Big(\sum_j |\varphi_j^*|^{p'} \Big)^{1/p'} \Bigg) \Bigg(\Big(\sum_j |\alpha_j|^p \frac{\big|Tx_j - Tx_j'\big|^p}{d(x_j, x_j')^p} \Big)^{1/p} \Bigg)$$

$$\leq \Bigg\| \Big(\sum_j |\varphi_j^*|^{p'} \Big)^{1/p'} \Bigg\|_{L^*} \sup_{\sum_j |\alpha_j|^p \leq 1} \Bigg\| \Big(\sum_j |\alpha_j|^p \frac{\big|Tx_j - Tx_j'\big|^p}{d(x_j, x_j')^p} \Big)^{1/p} \Bigg\|_E$$

The proof

The Lipschitz p-convexity of T allows us to bound this by

$$\left\| \left(\sum_j |\varphi_j^*|^{p'} \right)^{1/p'} \right\|_{E^*} M_{\mathrm{Lip}}^{(p)}(T) \sup_{\sum_j |\alpha_j|^p \leq 1} \left(\sum_j |\alpha_j|^p \frac{d(x_j, x_j')^p}{d(x_j, x_j')^p} \right)^{1/p}$$

$$= M_{\mathrm{Lip}}^{(p)}(T) \left\| \left(\sum_j |\varphi_j^*|^{p'} \right)^{1/p'} \right\|_{E^*}.$$

Therefore,

$$\left(\sum_j \left| \frac{(\hat{T}^* \varphi_j^*)(x_j) - (\hat{T}^* \varphi_j^*)(x_j')}{d(x_j, x_j')} \right|^{p'} \right)^{1/p'} \leq M_{\mathrm{Lip}}^{(p)}(T) \left\| \left(\sum_j |\varphi_j^*|^{p'} \right)^{1/p'} \right\|_{E^*},$$

so taking the supremum over all pairs $x_j, x_j' \in X$ with $x_j \neq x_j'$ we conclude

$$\left(\sum_j \left\| \hat{T}^* \varphi_j^* \right\|_{\mathrm{Lip}}^{p'} \right)^{1/p'} \leq M_{\mathrm{Lip}}^{(p)}(T) \left\| \left(\sum_j |\varphi_j^*|^{p'} \right)^{1/p'} \right\|_{E^*}.$$

The proof

Since the $\varphi_j^* \in L^*$ were arbitrary, this means that $\hat{T}^* : L^* \to \mathrm{Lip}_0(X)$ is p'-concave with $M_{(p')}(\hat{T}^*) \leq M_{\mathrm{Lip}}^{(p)}(T)$, and by duality $\hat{T} : \mathscr{F}(X) \to L$ is p-convex with $M^{(p)}(\hat{T}) \leq M_{\mathrm{Lip}}^{(p)}(T)$.

Remarks

- For a moment, one could think that in particular we have a result in the spirit of the Godefroy/Kalton theorem for the BAP, that is, for a Banach lattice E

$$E \text{ is } p\text{-convex} \quad \Leftrightarrow \quad \mathscr{F}(E) \text{ is } p\text{-convex.}$$

  However, what we do have is

$$id_E : E \to E \text{ is } p\text{-convex} \quad \Leftrightarrow \quad \widehat{id_E} : \mathscr{F}(E) \to E \text{ is } p\text{-convex.}$$

- Because of the role played by duality, it seems unlikely that these ideas could be used to prove a more similar result for other classes of operators obtained by replacing the expression $\left(\sum_j |x_j|^p\right)^{1/p}$ by other homogeneous functions given by the Krivine functional calculus for Banach lattices.

Linear p-concave maps

A linear operator $S : L \to E$ from a Banach lattice L to a Banach space E is called p-concave if there exists a constant $M < \infty$ such that for all $v_1, \ldots, v_n \in L$

$$\Big(\sum_{j=1}^{n} \|Sv_j\|_E^p \Big)^{1/p} \leq M \left\| \Big(\sum_{j=1}^{n} |v_j|^p \Big)^{1/p} \right\|_L , \qquad \text{if } 1 \leq p < \infty$$

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The smallest such constant M is denoted $M_{(p)}(T)$.

Lipschitz p-concave maps

Let X be a metric space and L a Banach lattice. A Lipschitz map $T : L \to X$ is called *Lipschitz p-concave* if there exists a constant $C \geq 0$ such that for any $v_j, v_j' \in L$,

$$\left(\sum_{j=1}^{n} d(Tv_j, Tv_j')^p \right)^{1/p} \leq C \left\| \left(\sum_{j=1}^{n} |v_j - v_j'|^p \right)^{1/p} \right\|_L .$$

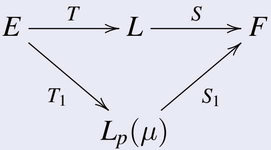
The smallest such constant C is the *Lipschitz p-concavity constant* of T and is denoted by $M_{(p)}^{\mathrm{Lip}}(T)$.

NOTE: In the case of Lipschitz concavity we can always “add constants” to the inequality.

Factorization through L_p

Theorem (Krivine)

Let E, F be Banach spaces and L a Banach lattice. Suppose that $T : E \to L$ is p-convex and $S : L \to F$ is p-concave. Then the operator ST can be factored through an $L_p(\mu)$ space. Moreover, we may arrange to have $ST = S_1 T_1$ with $T_1 : E \to L_p(\mu)$, $S_1 : L_p(\mu) \to F$, $\|T_1\| \le M^{(p)}(T)$ and $\|S_1\| \le M_{(p)}(S)$.



Can we get something nonlinear easily?

Suppose that $T : X \to L$ is Lipschitz p-convex and $S : L \to Y$ is Lipschitz p-concave.

$$X \xrightarrow{T} L \xrightarrow{S} Y$$

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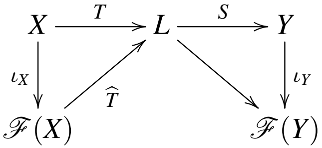

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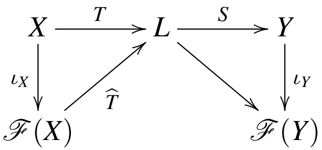
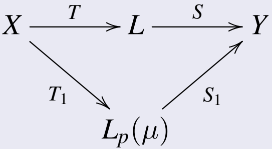


The map $\iota_Y \circ S$ is not linear!

Nonlinear factorization through L_p

Theorem (C)

Let X, Y be metric spaces with Y complete and L a Banach lattice. Suppose that $T : X \to L$ is Lipschitz p-convex and $S : L \to Y$ is Lipschitz p-concave. Then the operator ST can be factorized through an $L_p(\mu)$ space. Moreover, we may arrange to have $ST = S_1 T_1$ with $T_1 : X \to L_p(\mu)$, $S_1 : L_p(\mu) \to Y$, $\mathrm{Lip}(T_1) \leq M^{(p)}_{\mathrm{Lip}}(T)$ and $\mathrm{Lip}(S_1) \leq M^{\mathrm{Lip}}_{(p)}(S)$.

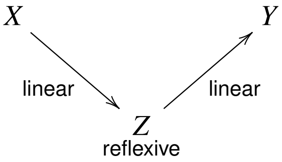


Magic?

- So far we have obtained nice results, but it's not quite clear why they work.
- The situation will be greatly clarified thanks to the factorization theory of Raynaud and Tradacete.

Factoring weakly compact operators

Easy way to construct a weakly compact operator: go through a reflexive space.



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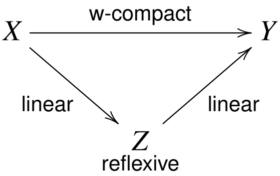

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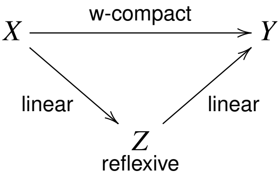
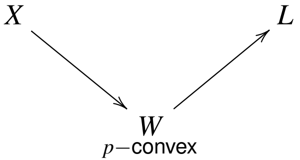


Theorem (Davis-Figiel-Johnson-Pelczynski)

This is the only way.

Factoring p-convex maps

Easy way to construct a p-convex linear map: go through a p-convex Banach lattice.



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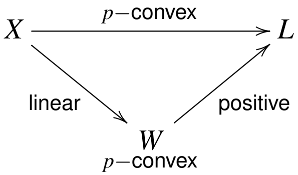

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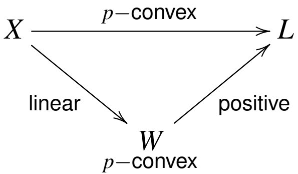


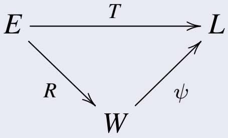
Theorem (Raynaud/Tradacete)

This is the only way.

Factoring p-convex maps

Theorem (Raynaud/Tradacete)

Let L be a Banach lattice, E a Banach space and $1 \leq p \leq \infty$. A linear operator $T : E \to L$ is p-convex if and only if there exist a p-convex Banach lattice W, a positive operator (in fact, an injective interval preserving lattice homomorphism) $\psi : W \to L$ and another linear operator $R : E \to W$ such that $T = \psi R$.

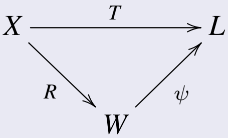


Moreover, $M^{(p)}(T) = \inf \|R\| \cdot M^{(p)}(I_W) \cdot \|\psi\|$.

Factorization characterization of Lipschitz p-convexity

Theorem (C)

Let L be a Banach lattice, X a metric space and $1 \leq p \leq \infty$. A Lipschitz map $T : X \to L$ is Lipschitz p-convex if and only if there exist a p-convex Banach lattice W, a positive operator (in fact, an injective interval preserving lattice homomorphism) $\psi : W \to L$ and another Lipschitz map $R : E \to W$ such that $T = \psi R$.



Moreover, $M_{\mathrm{Lip}}^{(p)}(T) = \inf \mathrm{Lip}(R) \cdot M^{(p)}(I_W) \cdot \|\psi\|$.

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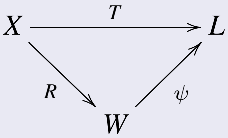
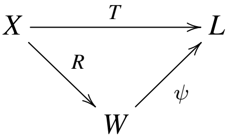


Moreover, $M^{(p)}_{\mathrm{Lip}}(T) = \inf \mathrm{Lip}(R) \cdot M^{(p)}(I_W) \cdot \|\psi\|$.

NOTE: This theorem implies the linear one.

A proof without duality

Suppose that $T : X \to L$ is Lipschitz p-convex.



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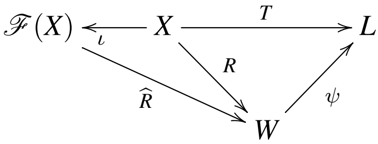

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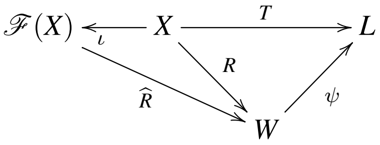
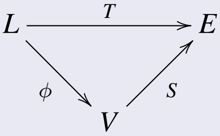


Hence $\widehat{T} = \psi \circ \widehat{R}$ is p-convex and with the same constant.

Factorization characterization of q-concavity

Theorem (Reisner; Raynaud/Tradacete)

Let L be a Banach lattice, E a Banach space and $1 \leq q \leq \infty$. A linear operator $T : L \to E$ is q-concave if and only if there exist a q-concave Banach lattice V, a positive operator $\phi : L \to V$ (in fact, a lattice homomorphism with dense image), and another operator $S : V \to E$ such that $T = S\psi$.

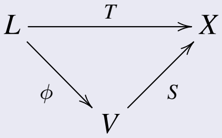


Moreover, $M_{(q)}(T) = \inf \|\phi\| \cdot M_{(q)}(I_V) \cdot \|S\|$.

Factorization characterization of Lipschitz q-concavity

Theorem (C)

Let L be a Banach lattice, X a complete metric space and $1 \leq q \leq \infty$. A Lipschitz map $T : L \to X$ is Lipschitz q-concave if and only if there exist a q-concave Banach lattice V, a positive operator $\phi : L \to V$ (in fact, a lattice homomorphism with dense image), and another Lipschitz map $S : V \to X$ such that $T = S\phi$.



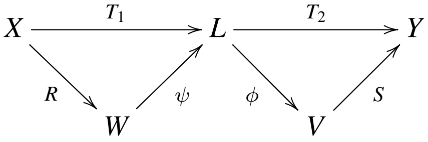
Moreover, $M^{\mathrm{Lip}}_{(q)}(T) = \inf \|\phi\| \cdot M_{(q)}(I_V) \cdot \mathrm{Lip}(S)$.

Nonlinear factorization through L_p revisited

Lemma (Raynaud/Tradacete)

If W, V are quasi-Banach lattices with W p-convex and V p-concave, then every lattice homomorphism $h : W \to V$ factors trough some $L_p(\mu)$, and the factors are lattice homomorphisms.

The nonlinear Krivine theorem is an easy consequence of the lemma and our characterizations: if $T_1 : X \to L$ is Lipschitz p-convex and $T_2 : L \to Y$ is Lipschitz p-concave (with Y complete),



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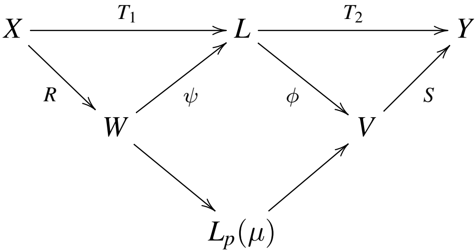
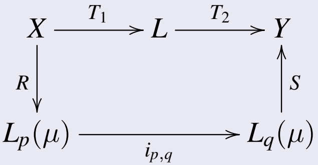


Nonlinear factorization through $L_p \to L_q$

Corollary (C)

Now if $T_1 : X \to L$ is Lipschitz p-convex and $T_2 : L \to Y$ is Lipschitz q-concave, with Y complete and $1 \leq p < q$, then $T_2 T_1$ Lipschitz factors through a canonical inclusion $i_{p,q} : L_p(\mu) \to L_q(\mu)$.



More related results

Proposition

Let X, Y be metric spaces with Y complete and E a Banach lattice, and $1 \leq p, q \leq \infty$. Suppose that $T : X \to E$ is Lipschitz p-convex and $S : E \to Y$ is Lipschitz q-concave. Then for every $\theta \in (0,1)$, ST factors through a Banach lattice U_θ that is $\frac{p}{p(1-\theta)+\theta}$-convex and $\frac{q}{1-\theta}$-concave.

Corollary

Let E be a Banach lattice, $1 \leq p, q \leq \infty$, and assume that $T : E \to E$ is both Lipschitz p-convex and Lipschitz q-concave. Then for each $\theta \in (0,1)$, T^2 factors through a $\frac{p}{p(1-\theta)+\theta}$-convex and $\frac{q}{1-\theta}$-concave Banach lattice. In particular, if $p > 1$ and $q < \infty$ then T factors through a super reflexive Banach lattice.

Lipschitz factorization implies linear factorization

If a linear map $T : X \to Y$ between Banach spaces can be factored as a Lipschitz p-convex map followed by a Lipschitz q-convex one, is there a factorization where the factor maps are in addition linear?

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Theorem (C)

Let $T : X \to Y$ be a linear map between a Banach space X and a dual Banach space Y, and assume that T admits a factorization $T = T_2 T_1$ where T_1 is Lipschitz p-convex and T_2 is Lipschitz q-concave, with $1 \le q < p < \infty$. Then there is also a factorization $T = \tau_2 \tau_1$ where τ_1 is p-convex and τ_2 is q-concave, and moreover $M^{(p)}(\tau_1) \le M^{(p)}_{\mathrm{Lip}}(T_1)$ and $M_{(q)}(\tau_2) \le M^{\mathrm{Lip}}_{(q)}(T_2)$.

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Remark: When $q = 2$ the theorem holds for a general Banach space Y, because in a Hilbert space all subspaces are 1-complemented.

Factorizations for just one map

What can we say if a linear operator $T : E \to F$ between Banach lattices is **both** p-convex and q-concave?

Does it factor as $T_2 T_1$ with T_2 p-convex and T_1 q-concave? (Note that such a product is always p-convex and q-concave)

$$\begin{array}{ccc} E & \xrightarrow{\;T\;} & F \\ {\scriptstyle \phi}\downarrow & & \uparrow{\scriptstyle \varphi} \\ E_q & \xrightarrow[\;R\;]{} & F_p \end{array}$$

where ϕ and φ are positive linear maps, E_q is q-concave, F_p is p-convex and R is a bounded linear map.

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Answer (Raynaud/Tradacete)

No. Example: for $1 < q < p < \infty$, the formal inclusion $L_p(0,1) \to L_q(0,1)$ does not admit such a factorization.

Nonlinear implies linear again

Theorem (C)

Let $T : E \to F$ be a linear map between Banach lattices E and F, and assume that T admits a factorization $T = T_2 T_1$ where T_1 is Lipschitz q-concave and T_2 is Lipschitz p-convex. Then there is also a factorization $T = \tau_2 \tau_1$ where τ_1 is q-concave and τ_2 is p-convex, and moreover $M^{(p)}(\tau_2) \leq M^{(p)}_{\mathrm{Lip}}(T_2)$ and $M_{(q)}(\tau_1) \leq M^{\mathrm{Lip}}_{(q)}(T_1)$.

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Corollary

A map that is both Lipschitz p-convex and Lipschitz q-concave does not necessarily admit a factorization as a Lipschitz q-concave map followed by a Lipschitz p-convex one.

"Giving up" a little

Theorem (Raynaud/Tradacete)

Suppose that a linear operator $T : E \to F$ *between Banach lattices is* p*-convex and* q*-concave, with* $1 < p \leq \infty$ *and* $1 \leq q < \infty$*. Then for every* $\theta \in (0,1)$*,* $T = T_2 T_1$ *where* T_2 *is* $p_\theta = \frac{p}{\theta + (1-\theta)p}$*-convex and* T_1 *is* $q_\theta = \frac{q}{1-\theta}$*-concave.*

“Giving up” a little

Theorem (Raynaud/Tradacete)

Suppose that a linear operator $T : E \to F$ between Banach lattices is p-convex and q-concave, with $1 < p \leq \infty$ and $1 \leq q < \infty$. Then for every $\theta \in (0,1)$, $T = T_2 T_1$ where T_2 is $p_\theta = \frac{p}{\theta + (1-\theta)p}$-convex and T_1 is $q_\theta = \frac{q}{1-\theta}$-concave. In fact, there is a factorization

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle \phi}\downarrow & & \uparrow{\scriptstyle \varphi} \\ E_\theta & \xrightarrow[R]{} & F_\theta \end{array}$$

where ϕ and φ are positive linear maps, E_θ is q_θ-concave, F_θ is p_θ-convex and R is a bounded linear map.

The nonlinear situation is unclear

Question

Suppose that a Lipschitz map $T : E \to F$ between Banach lattices is Lipschitz p-convex and Lipschitz q-concave, with $1 < p \leq \infty$ and $1 \leq q < \infty$. Can we find $1 < p_0 < p$ and $q < q_0 < \infty$ so that there is a factorization of T as

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ \phi \downarrow & & \uparrow \varphi \\ E_0 & \xrightarrow[R]{} & F_0 \end{array}$$

where ϕ and φ are positive linear maps, E_0 is q_0-concave, F_0 is p_0-convex and R is a Lipschitz map? Moreover: given $\theta \in (0, 1)$, can we have $p_0 = \frac{p}{\theta+(1-\theta)p}$ and $q_0 = \frac{q}{1-\theta}$?

Challenges

1. The proof of the linear result cannot be easily adapted to the Lipschitz context.
2. The arguments in that proof are heavily based on complex interpolation because that method works very well for lattices.
3. Complex interpolation, however, is not well suited to work with Lipschitz maps.
4. The results available require strong extra assumptions due to the fact that a Lipschitz function generally does not preserve analyticity.