The PDEs of Mathematical Finance

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Problems involving fair prices for stock options, bonds, etc, start with stochastic analysis and in some cases lead to deterministic parabolic equations of the form

$$\frac{\partial u}{\partial t} = \alpha x^k \frac{\partial^2 u}{\partial x^2} + (\beta + \gamma x) \frac{\partial u}{\partial x} + (\delta + \varepsilon x) u$$

for $t > 0$, $x \in J$. Here the Greek letters denote real constants with $\alpha > 0$. The generalized heat equation corresponds to $J = \mathbb{R}$, $k = 0$, $\gamma = \varepsilon = 0$; the generalized Black-Scholes equation corresponds to $J = (0, \infty)$, $k = 2$, $\beta = \varepsilon = 0$. The generalized Cox-Ingersoll-Ross equation corresponds to $J = (0, \infty)$, $k = 1$, $\delta = 0$, and β, γ both nonzero. These are deterministic equations having stochastic backgrounds in the mathematical finance context.

There is an natural correspondence between the heat equation

$$\frac{\partial v}{\partial t} = \widetilde{\alpha} \frac{\partial^2 v}{\partial x^2} + \widetilde{\beta} \frac{\partial v}{\partial x} + \widetilde{\gamma} v \tag{1}$$

for $x \in \mathbb{R}$ and $t > 0$ and the Black-Scholes equation with $k = 2$ and $\gamma = 0$ for various choics of constants with α, $\widetilde{\alpha}$ positive. We prove that the (C_0) semigroups governing these problems are intimately connected and are chaotic on certain weighted sup norm spaces with various positive weights w,

$$Y_w = \{f \in C(J) : wf \in C_0(J)\}.$$

Results to be presented include semigroup generation, chaos for the generalized heat and Black-Scholes equations, and a new Feynman-Kac type formula for the Cox-Ingersoll-Ross equation. In addition, recent extensions to time dependent volatility and option rates, for both the European and Asian stock option models for the Black-Scholes equation will be presented. This work is joint with H. Emamirad, J. A. Goldstein, M.Kaplan, R. Mininni, Ph. Rogeon, and S. Romanelli.