Polynomials on Banach spaces

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1. Multilinear mappings and polynomials

In this section we introduce the notions of multilinear mappings and polynomials between normed linear spaces, and prove some of their main properties. In particular, homogeneous polynomials are in a canonical one-to-one correspondence with the symmetric multilinear forms via the Polarisation formula. Polynomials are continuous mappings whenever they have at least one point of continuity. We introduce non-homogeneous polynomials of degree at most n and prove that their space is a direct sum of the homogeneous summands.

Definition 1. Let $X_1, \dots, X_n, Y$ be vector spaces. We say that a mapping $M\colon X_1 \times \cdots \times X_n \to Y$ is n-linear if it is linear in each coordinate, i.e. $x \mapsto M(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n)$ is a linear mapping from X_k into Y for each $x_1 \in X_1, \dots, x_n \in X_n$ and each $k \in \{1, \dots, n\}$. By $L(X_1, \dots, X_n; Y)$ we denote the vector space of all n-linear mappings from $X_1 \times \cdots \times X_n$ to Y. In the special case when $X_k = X$, $1 \le k \le n$, we use the short notation $L({}^nX; Y)$. A mapping is called multilinear if it is n-linear for some $n \in \mathbb{N}$. A 2-linear mapping will also be called bilinear.

We say that $M \in L({}^nX; Y)$ is symmetric if $M(x_1, \dots, x_n) = M(x_{\pi(1)}, \dots, x_{\pi(n)})$ for every permutation π of $\{1, \dots, n\}$ and every $x_1, \dots, x_n \in X$. By $L^{\mathrm{s}}({}^nX; Y)$ we denote the vector space of all n-linear symmetric mappings from X^n to Y.

By repeated applications of the definition we can see that an n-linear mapping $M \in L(X_1, \dots, X_n; Y)$ satisfies the formula

$$M\left(\sum_{k_1=1}^{m_1} a^1_{k_1} x^1_{k_1}, \dots, \sum_{k_n=1}^{m_n} a^n_{k_n} x^n_{k_n}\right) = \sum_{\substack{1 \le k_j \le m_j \\ j=1,\dots,n}} \left(\prod_{j=1}^{n} a^j_{k_j}\right) M(x^1_{k_1}, \dots, x^n_{k_n}) \tag{1}$$

for an arbitrary choice of $m_j \in \mathbb{N}$, $a^j_{k_j} \in \mathbb{K}$, and $x^j_{k_j} \in X_j$, where $k_j = 1, \dots, m_j$ and $j = 1, \dots, n$.

Definition 2. Let $X_1, \dots, X_n, Y$ be normed linear spaces. We say that $M \in L(X_1, \dots, X_n; Y)$ is a bounded n-linear mapping, if

$$\|M\| = \sup_{x_1 \in B_{X_1}, \dots, x_n \in B_{X_n}} \|M(x_1, \dots, x_n)\| < +\infty.$$

It is easily checked that this defines a norm on the subspace of $L(X_1, \dots, X_n; Y)$ consisting of bounded multilinear mappings. By $\bigl(\mathcal{L}(X_1, \dots, X_n; Y), \|\cdot\|\bigr)$, resp. $\bigl(\mathcal{L}({}^nX; Y), \|\cdot\|\bigr)$, resp. $\bigl(\mathcal{L}^{\mathrm{s}}({}^nX; Y), \|\cdot\|\bigr)$, we denote the normed linear spaces of all respective n-linear bounded mappings. For bounded n-linear forms, i.e. when the target space is the scalar field, we use a shortened notation $\mathcal{L}({}^nX) = \mathcal{L}({}^nX; \mathbb{K})$.

Let $M \in \mathcal{L}(X_1, \dots, X_n; Y)$. Then by the homogeneity we have

$$\|M(x_1, \dots, x_n)\| \le \|M\|\|x_1\| \cdots \|x_n\| \quad \text{for } x_j \in X_j,\ j = 1, \dots, n. \tag{2}$$

Proposition 3. *Let $X_1, \dots, X_n$, Y be normed linear spaces and $M \in L(X_1, \dots, X_n; Y)$. The following statements are equivalent:*

(i) *M is bounded.*
(ii) *M is Lipschitz on bounded sets.*
(iii) *M is continuous.*
(iv) *M is bounded on a neighbourhood of some point.*

PROOF. We consider the maximum norm on the space $X_1 \oplus \cdots \oplus X_n$. For $x \in X_1 \oplus \cdots \oplus X_n$ we write $x = (x_j)_{j=1}^n$.

(i)⇒(ii) Let $S \subset X_1 \oplus \cdots \oplus X_n$ be a given bounded set and let $c > 0$ be such that $S \subset (cB_{X_1}) \times \cdots \times (cB_{X_n})$. Using (1), for any $x, y \in S$ we obtain

$$M(y) = M(x + y - x) = M(x) + \sum_{A \subsetneq \{1,\dots,n\}} M\Bigl(\bigl(\chi_A(j)x_j + (1 - \chi_A(j))(y_j - x_j)\bigr)_{j=1}^n\Bigr).$$

Combining this with (2) we get

$$\|M(y) - M(x)\| \le 2^n \|M\| (2c)^{n-1} \max_{1 \le j \le n} \|y_j - x_j\|. \tag{3}$$

(ii)⇒(iii)⇒(iv) is clear, so it remains to show (iv)⇒(i). Using (1), for any $x, h \in X_1 \oplus \cdots \oplus X_n$ we obtain

$$\begin{aligned} M(h) &= \sum_{A \subset \{1,\ldots,n\}} M\Big(\big(\chi_A(j)(x_j + h_j) + (1 - \chi_A(j))(-x_j)\big)_{j=1}^n\Big) \\ &= \sum_{A \subset \{1,\ldots,n\}} (-1)^{n-|A|} M\Big(\big(\chi_A(j)(x_j + h_j) + (1 - \chi_A(j))x_j\big)_{j=1}^n\Big). \end{aligned}$$

It follows that if M is bounded on a neighbourhood of x, then it is also bounded on a neighbourhood of zero, and hence by the homogeneity also on the unit ball.

□

Let X, Y be vector spaces and $M \in L(^nX; Y)$. We define the symmetrisation of M by

$$M^{\mathrm{s}}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\pi \in S_n} M(x_{\pi(1)}, \ldots, x_{\pi(n)}),$$

where S_n is the set of all permutations of $\{1, \ldots, n\}$. Then $M^{\mathrm{s}} \in L^{\mathrm{s}}(^nX; Y)$. Furthermore, if X and Y are normed linear spaces and M is bounded, then so is M^{s} and obviously $\|M^{\mathrm{s}}\| \le \|M\|$. If M is symmetric, then clearly $M^{\mathrm{s}} = M$.

Proposition 4 (Polarisation formula)**.** *Let X, Y be vector spaces and $M \in L(^nX; Y)$. Then*

$$M^{\mathrm{s}}(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n M\left(a + \sum_{j=1}^n \varepsilon_j x_j, \ldots, a + \sum_{j=1}^n \varepsilon_j x_j\right)$$

for every $a, x_1, \ldots, x_n \in X$. In particular, if M is symmetric, then it is uniquely determined by its values $M(x, \ldots, x)$, $x \in X$, along the diagonal.

PROOF. For convenience we put $x_0 = a$ and $\varepsilon_0 = 1$. Using (1) we obtain

$$\begin{aligned} &\frac{1}{2^n n!} \sum_{\substack{\varepsilon_j = \pm 1 \\ j=1,\ldots,n}} \varepsilon_1 \cdots \varepsilon_n M\left(\sum_{j=0}^n \varepsilon_j x_j, \ldots, \sum_{j=0}^n \varepsilon_j x_j\right) \\ &\qquad = \frac{1}{2^n n!} \sum_{\substack{\varepsilon_j = \pm 1 \\ j=1,\ldots,n}} \varepsilon_1 \cdots \varepsilon_n \sum_{j_1,\ldots,j_n \in \{0,\ldots,n\}} \varepsilon_{j_1} \cdots \varepsilon_{j_n} M(x_{j_1}, \ldots, x_{j_n}) \\ &\qquad = \frac{1}{2^n n!} \sum_{j_1,\ldots,j_n \in \{0,\ldots,n\}} \left(\sum_{\substack{\varepsilon_j = \pm 1 \\ j=1,\ldots,n}} \varepsilon_1 \cdots \varepsilon_n \varepsilon_{j_1} \cdots \varepsilon_{j_n}\right) M(x_{j_1}, \ldots, x_{j_n}). \end{aligned}$$

If there is $k \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_n\}$, then

$$\sum_{\substack{\varepsilon_j = \pm 1 \\ j=1,\ldots,n}} \varepsilon_1 \cdots \varepsilon_n \varepsilon_{j_1} \cdots \varepsilon_{j_n} = \sum_{\substack{\varepsilon_j = \pm 1 \\ j \ne k}} \varepsilon_{j_1} \cdots \varepsilon_{j_n} \prod_{j \ne k} \varepsilon_j + (-1) \cdot \sum_{\substack{\varepsilon_j = \pm 1 \\ j \ne k}} \varepsilon_{j_1} \cdots \varepsilon_{j_n} \prod_{j \ne k} \varepsilon_j = 0. \tag{4}$$

It follows that

$$\begin{aligned}\frac{1}{2^n n!}\sum_{j_1,\dots,j_n\in\{0,\dots,n\}}&\left(\sum_{\substack{\varepsilon_j=\pm1\\ j=1,\dots,n}}\varepsilon_1\cdots\varepsilon_n\varepsilon_{j_1}\cdots\varepsilon_{j_n}\right)M(x_{j_1},\dots,x_{j_n})\\ &=\frac{1}{2^n n!}\sum_{\pi\in S_n}\left(\sum_{\varepsilon_j=\pm1}\varepsilon_1^2\cdots\varepsilon_n^2\right)M(x_{\pi(1)},\dots,x_{\pi(n)})\\ &=\frac{1}{n!}\sum_{\pi\in S_n}M(x_{\pi(1)},\dots,x_{\pi(n)})=M^{\mathrm{s}}(x_1,\dots,x_n),\end{aligned}$$

where S_n is the set of all permutations of $\{1,\dots,n\}$. This finishes the proof.

□

Definition 5. Let X, Y be vector spaces and $n\in\mathbb{N}$. A mapping $P\colon X\to Y$ is said to be an n-homogeneous polynomial if there exists an n-linear mapping $M\in L(^nX;Y)$ such that $P(x)=M(x,\dots,x)$. We use the notation $P=\widehat{M}$. For convenience we also define 0-homogeneous polynomials as constant mappings from X to Y. We denote by $P(^nX;Y)$, $n\in\mathbb{N}_0$, the vector space of all n-homogeneous polynomials from X into Y.

Suppose X, Y are normed linear spaces, $n\in\mathbb{N}_0$. We say that $P\in P(^nX;Y)$ is a bounded polynomial, if

$$\|P\|=\sup_{x\in B_X}\|P(x)\|<+\infty.$$

We denote by $\left(\mathcal{P}(^nX;Y),\|\cdot\|\right)$ the normed linear space of all n-homogeneous bounded polynomials from X into Y. When the target space is the scalar field, we use a shortened notation $\mathcal{P}(^nX)=\mathcal{P}(^nX;\mathbb{K})$.

For a given n-homogeneous polynomial P the n-linear mapping M that gives rise to P is not determined uniquely. In particular, the symmetrised n-linear mapping leads to the same polynomial: for every $M\in L(^nX;Y)$ we have $\widehat{M}=\widehat{M^{\mathrm{s}}}$. However, the following fundamental result holds.

Proposition 6 (Polarisation formula; [**BohHil:DirichletSer**], [**MazOrl:Polynom1**])**.** *Let X, Y be vector spaces and $n\in\mathbb{N}$. For every $P\in P(^nX;Y)$ there exists a unique symmetric n-linear mapping $\check{P}\in L^{\mathrm{s}}(^nX;Y)$ such that $P(x)=\check{P}(x,\dots,x)$. It satisfies the formula*

$$\check{P}(x_1,\dots,x_n)=\frac{1}{2^n n!}\sum_{\varepsilon_j=\pm1}\varepsilon_1\cdots\varepsilon_n P\left(a+\sum_{j=1}^n\varepsilon_jx_j\right),\tag{5}$$

where $a\in X$ can be chosen arbitrarily. Moreover, if X, Y are normed linear spaces and P is bounded, then $\check{P}$ is also bounded and we have

$$\|P\|\le\|\check{P}\|\le\frac{n^n}{n!}\|P\|.\tag{6}$$

On the other hand, for every $m>n$ and $a,x_1,\dots,x_m\in X$ the following holds:

$$\sum_{\varepsilon_j=\pm1}\varepsilon_1\cdots\varepsilon_m P\left(a+\sum_{j=1}^m\varepsilon_jx_j\right)=0.\tag{7}$$

Proof. Let $M\in L(^nX;Y)$ be such that $P=\widehat{M}$. Put $\check{P}=M^{\mathrm{s}}$. Obviously, $P=\widehat{M^{\mathrm{s}}}$ and so using Proposition 4 we obtain

$$\begin{aligned}\check{P}(x_1,\dots,x_n)=M^{\mathrm{s}}(x_1,\dots,x_n)&=\frac{1}{2^n n!}\sum_{\varepsilon_j=\pm1}\varepsilon_1\cdots\varepsilon_n M^{\mathrm{s}}\left(a+\sum_{j=1}^n\varepsilon_jx_j,\dots,a+\sum_{j=1}^n\varepsilon_jx_j\right)\\ &=\frac{1}{2^n n!}\sum_{\varepsilon_j=\pm1}\varepsilon_1\cdots\varepsilon_n P\left(a+\sum_{j=1}^n\varepsilon_jx_j\right).\end{aligned}$$

The uniqueness follows from the fact that a symmetric n-linear mapping is uniquely determined by its values along the diagonal (Proposition 4). The estimate (6) follows readily from the formula (5).

To prove (7) it is enough to follow the proof of Proposition 4 and notice that for each summand there is always $k \in \{1, \dots, m\} \setminus \{j_1, \dots, j_n\}$ and so (4) holds.

$\square$

Definition 7. Let $n \in \mathbb{N}$. For a multi-index $\alpha \in \mathbb{N}_0^n$ we denote its order by $|\alpha| = \sum_{j=1}^n \alpha_j$. Further, we denote the set of multi-indices of order $d \in \mathbb{N}_0$ by

$$\mathcal{I}(n, d) = \big\{\alpha \in \{0, \dots, d\}^n;\ |\alpha| = d\big\}.$$

We extend the definition also to the case when $n = \infty$, setting

$$\mathcal{I}(\infty, d) = \left\{\alpha \in \{0, \dots, d\}^{\mathbb{N}};\ |\alpha| = \sum_{j=1}^{\infty} \alpha_j = d\right\}.$$

For $n \in \mathbb{N}$ we have $|\mathcal{I}(n, d)| = \binom{n+d-1}{n-1}$ (indeed, from the combinatorial point of view it represents the number of distributions of d identical balls into n distinct boxes). A given $(k_j)_{j=1}^d \in \{1, \dots, n\}^d$ determines a unique $\alpha \in \mathcal{I}(n, d)$ by the relation

$$\alpha = \big(|\{j;\ k_j = 1\}|, |\{j;\ k_j = 2\}|, \dots, |\{j;\ k_j = n\}|\big). \tag{8}$$

Conversely, a given $\alpha \in \mathcal{I}(n, d)$ determines a unique $k(\alpha) = \big(k_1(\alpha), \dots, k_d(\alpha)\big)$, $k_1(\alpha) \le \dots \le k_d(\alpha)$, such that (8) holds. Given $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{I}(n, d)$ we use the standard multi-index notation

$$x^\alpha = \prod_{l=1}^{n} x_l^{\alpha_l} = \prod_{j=1}^{d} x_{k_j(\alpha)}.$$

The case $n = \infty$ is similar and corresponds to multi-indices whose domain is $\mathbb{N}$.

Note that $x \mapsto x^\alpha \in \mathcal{P}(^d\mathbb{K}^n)$ for any $\alpha \in \mathcal{I}(n, d)$. Indeed, denote by $\phi_l \in (\mathbb{K}^n)^*$ the coordinate functional $\phi_l(x) = x_l$. Let $M \in \mathcal{L}(^d\mathbb{K}^n; \mathbb{K})$ be given by the formula $M(y_1, \dots, y_d) = \prod_{j=1}^d \phi_{k_j(\alpha)}(y_j)$. Then we have $x^\alpha = \widehat{M}(x)$.

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{I}(n, d)$, we denote the corresponding multinomial coefficient by

$$\binom{d}{\alpha} = \binom{d}{\alpha_1, \dots, \alpha_n} = \frac{d!}{\alpha!} = \frac{d!}{\alpha_1! \cdots \alpha_n!}.$$

Proposition 8 (Multinomial formula)**.** *Let X, Y be vector spaces, $d \in \mathbb{N}$, $P \in P(^dX; Y)$, and $x_1, \dots, x_n \in X$. Then*

$$P(x_1 + \dots + x_n) = \sum_{\alpha \in \mathcal{I}(n,d)} \binom{d}{\alpha} \check{P}(^{\alpha_1}x_1, \dots, {}^{\alpha_n}x_n).$$

PROOF. Using (1) and then collecting the terms according to the value of α using the symmetry of $\check{P}$ we obtain

$$P\left(\sum_{j=1}^{n} x_j\right) = \sum_{\substack{1 \le k_j \le n \\ j=1,\dots,d}} \check{P}(x_{k_1}, \dots, x_{k_d}) = \sum_{\alpha \in \mathcal{I}(n,d)} \binom{d}{\alpha} \check{P}(^{\alpha_1}x_1, \dots, {}^{\alpha_n}x_n).$$

$\square$

The next proposition asserts that the abstract definition of homogeneous polynomials coincides on $\mathbb{K}^n$ with the classical definition that uses coordinates. Note that in this case all homogeneous polynomials are automatically bounded.

Proposition 9. *Let $n, d \in \mathbb{N}$ and Y be a vector space over $\mathbb{K}$. A mapping $P : \mathbb{K}^n \to Y$ is a d-homogeneous polynomial if and only if there exist $\{y_\alpha\}_{\alpha\in\mathcal{I}(n,d)} \subset Y$ such that $P(x) = \sum_{\alpha\in\mathcal{I}(n,d)} x^\alpha y_\alpha$. Moreover, each y_α is uniquely determined by*

$$y_\alpha = \binom{d}{\alpha} \check{P}({}^{\alpha_1}e_1, \dots, {}^{\alpha_n}e_n),$$

where $\{e_j\}_{j=1}^n$ is the canonical basis of $\mathbb{K}^n$.

PROOF. Let $P \in \mathcal{P}({}^d\mathbb{K}^n; Y)$. By Proposition 8 we have

$$P\left(\sum_{j=1}^n x_j e_j\right) = \sum_{\alpha\in\mathcal{I}(n,d)} x^\alpha \binom{d}{\alpha} \check{P}({}^{\alpha_1}e_1, \dots, {}^{\alpha_n}e_n).$$

The converse is clear from the fact that $x \mapsto x^\alpha \in \mathcal{P}({}^d\mathbb{K}^n)$.

To prove the uniqueness we show somewhat more: if $Q(x_1, \dots, x_n) = \sum_{\alpha\in A} x^\alpha z_\alpha = 0$ for each $x \in \mathbb{K}^n$, then $z_\alpha = 0$ for each $\alpha \in A$, where $A = \{\alpha \in \mathbb{N}_0^n;\ |\alpha| \le d\}$. First notice that it suffices to show this only for $Y = \mathbb{K}$, the general case follows by using composition with linear functionals from Y^*. We use induction on n. For $n = 1$ it is a standard fact that can be proved for example by factorisation. To prove the induction step from $n-1$ to n put $A_k = \{\alpha \in A;\ \alpha_1 = k\}$, $k = 0, \dots, d$. Then $Q(x_1, \dots, x_n) = \sum_{k=0}^d x_1^k \sum_{\alpha\in A_k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} z_\alpha = 0$ for each $x_1, \dots, x_n \in \mathbb{K}$. From the first step of the induction it follows that $\sum_{\alpha\in A_k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} z_\alpha = 0$ for each $x_2, \dots, x_n \in \mathbb{K}$ and $k \in \{0, \dots, d\}$. Hence by the inductive hypothesis $z_\alpha = 0$ for every $\alpha \in \bigcup_{k=0}^d A_k = A$.

□

In the special case $Y = \mathbb{K}$ this reduces to the familiar formula $P(x) = \sum_{\alpha\in\mathcal{I}(n,d)} a_\alpha x^\alpha$, where the coefficients $a_\alpha \in \mathbb{K}$.

While working with polynomials $P \in P({}^d\mathbb{K}^n; Y)$ we are often going to use freely the classical notation using the coordinates and multinomial expressions, e.g. we will sometimes write $P(x_1, \dots, x_n)$ instead of the formally correct $P\big((x_1, \dots, x_n)\big)$, $x_j \in \mathbb{K}$, $j = 1, \dots, n$.

Proposition 10. *Let X be a normed linear space with a Schauder basis $\{e_j\}_{j=1}^\infty$, Y a vector space, $d \in \mathbb{N}$, and $P \in P({}^dX; Y)$. Denote $X_0 = \operatorname{span}\{e_j\}_{j=1}^\infty$. Then there is a unique collection of vectors $\{y_\alpha\}_{\alpha\in\mathcal{I}(\infty,d)} \subset Y$ such that the formula*

$$P(x) = \sum_{\alpha\in\mathcal{I}(\infty,d)} x^\alpha y_\alpha \tag{9}$$

holds for every $x \in X_0$. The coefficients y_α are given by

$$y_\alpha = \binom{n}{\alpha} \check{P}({}^{\alpha_1}e_1, {}^{\alpha_2}e_2, \dots).$$

Conversely, any $\{y_\alpha\}_{\alpha\in\mathcal{I}(\infty,d)} \subset Y$ uniquely determines a polynomial $P \in P({}^dX_0; Y)$ by the formula (9).

Definition 11. Let X, Y be vector spaces and $n \in \mathbb{N}_0$. A mapping $P : X \to Y$ is called a polynomial of degree at most n if there are $P_k \in P({}^kX; Y)$, $k = 0, \dots, n$, such that $P = \sum_{k=0}^n P_k$. If $P_n \ne 0$, we say that P has degree n and we use the notation $\deg P = n$. We denote by $P^n(X; Y)$ the space of all polynomials of degree at most n. We denote by $P(X; Y) = \bigcup_{n=0}^\infty P^n(X; Y)$ the space of all polynomials.

The fact that $\deg P$ is well-defined will be apparent shortly from Corollary 12.

Corollary 12. *Let X, Y be vector spaces, $n \in \mathbb{N}_0$, and let $P \in P^n(X; Y)$ be such that $P = \sum_{k=0}^n P_k$, $P_k \in P({}^kX; Y)$. Then the homogeneous summands P_k of P are uniquely determined.*

2. Complexification

Definition 13. By a complexification of a real normed linear space $(X, \|\cdot\|)$ we mean the complex normed linear space $\tilde{X} = \{(x, y);\ x, y \in X\}$ with operations

$$(x, y) + (u, v) = (x + u, y + v),$$
$$(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x),$$

and a norm $\|\cdot\|_{\mathbb{C}}$ defined as

$$\|(x, y)\|_{\mathbb{C}} = \sup_{\phi \in B_{X^*}} \sqrt{\phi(x)^2 + \phi(y)^2}.$$

Checking that $(\tilde{X}, \|\cdot\|_{\mathbb{C}})$ is a complex normed linear space real-isomorphic to $X \oplus X$ is easy.

Given real normed linear spaces X, Y, and given $T \in \mathcal{L}(X; Y)$, it is easily checked that there is a uniquely determined (by complex linearity) complex extension $\tilde{T} \in \mathcal{L}(\tilde{X}; \tilde{Y})$, defined by $\tilde{T}(x + iy) = T(x) + iT(y)$.

Proposition 14. *Let X, Y be real normed linear spaces and $T \in \mathcal{L}(X; Y)$. Then $\|\tilde{T}\|_{\mathbb{C}} = \|T\|$.*

PROOF. We only need to verify that $\|\tilde{T}\|_{\mathbb{C}} \leq \|T\|$:

$$\begin{aligned}\|\tilde{T}(x + iy)\|_{\mathbb{C}} &= \|T(x) + iT(y)\| = \sup_{t \in [0, 2\pi]} \|\cos(t)T(x) + \sin(t)T(y)\| \\ &= \sup_{t \in [0, 2\pi]} \|T(\cos(t)x + \sin(t)y)\| \leq \|T\| \sup_{t \in [0, 2\pi]} \|\cos(t)x + \sin(t)y\| = \|T\| \|x + iy\|_{\mathbb{C}}.\end{aligned}$$

□

Obviously, T is one-to-one if and only if $\tilde{T}$ is one-to-one, and T is onto if and only if $\tilde{T}$ is onto.

Proposition 15 ([**BochSic:Poly**]). *Let X, Y be real normed linear spaces, $n \in \mathbb{N}$, and $M \in L(^nX; Y)$. Then there is a unique extension $\widetilde{M} \in L(^n\tilde{X}; \tilde{Y})$ such that $\widetilde{M}\restriction_{X^n} = M$. It is given by the formula*

$$\widetilde{M}(x_1^0 + ix_1^1, \dots, x_n^0 + ix_n^1) = \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{0,1\}} i^{\sum_{j=1}^n \varepsilon_j} M(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n}). \tag{10}$$

Moreover, $\widetilde{M}$ is bounded if and only if M is bounded.

Proposition 16 ([**Tay:Poly**]). *Let X, Y be real normed linear spaces, $n \in \mathbb{N}$, and $P \in P(^nX; Y)$. Then there is a unique extension $\tilde{P} \in P(^n\tilde{X}; \tilde{Y})$ such that $\tilde{P}\restriction_X = P$. It is given by the formula*

$$\tilde{P}(x + iy) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} \check{P}(^{n-2k}x, {}^{2k}y) + i \sum_{k=0}^{[\frac{(n-1)}{2}]} (-1)^k \binom{n}{2k+1} \check{P}(^{n-(2k+1)}x, {}^{2k+1}y).$$

Moreover, $\tilde{P}$ is bounded if and only if P is bounded.

For $P \in \mathcal{P}(^0X; Y)$ the extension is trivial. Naturally, if $P(x) = \sum_{k=0}^n P_k(x)$, $P_k \in P(^kX; Y)$, then we denote $\tilde{P}(x) = \sum_{k=0}^n \widetilde{P_k}(x)$.

Theorem 17 ([**MuSaTo:Complex**]). *Let X, Y be real normed linear spaces and $P \in \mathcal{P}^n(X; Y)$. Then*

$$\|\tilde{P}\|_{\mathbb{C}} \leq 2^{\frac{n}{2}} |T_n(i)| \cdot \|P\|.$$

3. Estimates of coefficients of polynomials

In this section we show some applications of a simple complex averaging technique which leads to many useful and sharp coefficient estimates.

We begin with a few result for complex polynomials analogous to the Polarisation formula. Let us define the generalised Rademacher system. For given $n, k \in \mathbb{N}$ let $r_1^{n,k}(l) = e^{i2\pi\frac{m}{n}}$ for $l \in \mathbb{N}$, $mn^{k-1} \leq$

$(l-1) \bmod n^k < (m+1)n^{k-1}$, $0 \le m \le n-1$, and put $r_j^{n,k}(l) = r_1^{n,k}(n^{j-1}l)$, $l \in \{1,\dots,n^k\}$, $j = 2,\dots,k$. We note that the system can be defined also on $[0,1]$ or more generally on a suitable probability space, but we chose to work with the discrete domain to underline the discrete nature of the methods used here.

Lemma 18 ([**ArLaRyTo:Rademacher**], [**Dineen:Complex**]). *Let $n, k \in \mathbb{N}$. The generalised Rademacher system has the following property: If $m_1, \dots, m_k \in \mathbb{N}_0$, then the average*

$$\frac{1}{n^k}\sum_{l=1}^{n^k} r_1^{n,k}(l)^{m_1}\cdots r_k^{n,k}(l)^{m_k} = \begin{cases} 1 & \text{if } m_j \equiv 0 \pmod n,\ j = 1,\dots,k, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The first case is clear, since $r_j^{n,k}(l)^n = 1$ for each $l \in \{1,\dots,n^k\}$, $j = 1,\dots,k$. To prove the other case let s be the biggest number in $\{1,\dots,k\}$ such that $m_s \not\equiv 0 \pmod n$. Then

$$\sum_{l=1}^{n^k} r_1^{n,k}(l)^{m_1}\cdots r_k^{n,k}(l)^{m_k} = \sum_{q=1}^{n^{s-1}}\left(r_1^{n,k}(qn^{k-s+1})^{m_1}\cdots r_{s-1}^{n,k}(qn^{k-s+1})^{m_{s-1}} \sum_{l=1}^{n^{k-s+1}} r_s^{n,k}(l)^{m_s}\right) = 0$$

since $\sum_{j=0}^{n-1} e^{i2\pi \frac{j}{n} m_s} = 0$.

□

The following two results should be compared with the Polarisation formula (Proposition 6, Lemma **??**), where $k = n$ and $\varepsilon_j = r_j^{2,n}$.

Proposition 19 ([**ArLaRyTo:Rademacher**]). *Let X, Y be complex vector spaces, $n \in \mathbb{N}$, and $P \in P({}^nX;Y)$. Then*

$$\check{P}({}^{n_1}x_1,\dots,{}^{n_k}x_k) = \frac{n_1!\cdots n_k!}{n!}\frac{1}{n^k}\sum_{l=1}^{n^k} r_1^{n,k}(l)^{n-n_1}\cdots r_k^{n,k}(l)^{n-n_k} P\left(\sum_{j=1}^{k} r_j^{n,k}(l)x_j\right)$$

for all $x_1,\dots,x_k \in X$, $n_1,\dots,n_k \in \mathbb{N}$ with $n_1+\dots+n_k = n$, $1 \le k \le n$.

PROOF. Without loss of generality we may assume that $k \ge 2$. Using the Multinomial formula (Proposition 8) and Lemma 18 we obtain

$$\begin{aligned}
&\frac{1}{n^k}\sum_{l=1}^{n^k} r_1^{n,k}(l)^{n-n_1}\cdots r_k^{n,k}(l)^{n-n_k} P\left(\sum_{j=1}^{k} r_j^{n,k}(l)x_j\right)\\
&\quad= \frac{1}{n^k}\sum_{l=1}^{n^k}\sum_{\alpha\in\mathcal{J}(k,n)}\binom{n}{\alpha} r_1^{n,k}(l)^{n-n_1+\alpha_1}\cdots r_k^{n,k}(l)^{n-n_k+\alpha_k}\check{P}({}^{\alpha_1}x_1,\dots,{}^{\alpha_k}x_k)\\
&\quad= \sum_{\alpha\in\mathcal{J}(k,n)}\binom{n}{\alpha}\check{P}({}^{\alpha_1}x_1,\dots,{}^{\alpha_k}x_k)\left(\frac{1}{n^k}\sum_{l=1}^{n^k} r_1^{n,k}(l)^{n-n_1+\alpha_1}\cdots r_k^{n,k}(l)^{n-n_k+\alpha_k}\right)\\
&\quad= \binom{n}{n_1,\dots,n_k}\check{P}({}^{n_1}x_1,\dots,{}^{n_k}x_k)
\end{aligned}$$

from which the formula follows.

□

Lemma 20. *Let X, Y be complex vector spaces, $n \in \mathbb{N}$, and let $P \in P^n(X;Y)$ be such that $P = \sum_{m=0}^{n} P_m$, $P_m \in P({}^mX;Y)$. Then for every $x_1,\dots,x_k \in X$*

$$P_0(0) + \sum_{j=1}^{k} P_n(x_j) = \frac{1}{n^k}\sum_{l=1}^{n^k} P\left(\sum_{j=1}^{k} r_j^{n,k}(l)x_j\right).$$

PROOF. By Lemma 18

$$\frac{1}{n^k}\sum_{l=1}^{n^k} P\left(\sum_{j=1}^{k} r_j^{n,k}(l)x_j\right) = P_0(0) + \sum_{m=1}^{n}\frac{1}{n^k}\sum_{l=1}^{n^k}\sum_{\substack{1\le j_s\le k\\ s=1,\dots,m}} r_{j_1}^{n,k}(l)\cdots r_{j_m}^{n,k}(l)\check{P}_m(x_{j_1},\dots,x_{j_m})$$
$$= P_0(0) + \sum_{j=1}^{k} P_n(x_j).$$

□

Theorem 21 ([**AroGlo:Analc0**], [**ArBeEn:PolyNorms**]). *Let $p \in \mathcal{P}^n(\mathbb{C}^N)$ and*

$$p(x_1,\dots,x_N) = \sum_{j=1}^{N} a_j x_j^n + \sum_{\substack{\beta\in\mathcal{J}(N,n)\\ \beta\neq(0,\dots,0,n,0,\dots,0)}} a_\beta x^\beta$$

Then

$$\sum_{j=1}^{N}|a_j| \le \max_{z_j\in\mathbb{C},|z_j|=1}|p(z_1,\dots,z_N)| \le \|p\|_\infty.$$

In particular, in the complex scalar case, if $P\in\mathcal{P}(^n c_0)$, then $\sum_{j=1}^{\infty}|P(e_j)| \le \|P\|_\infty$.

PROOF. Assume without loss of generality that the constant term C of p is real and non-negative. Let $\lambda_j\in\mathbb{C}$ be such that $|\lambda_j| = 1$ and $a_j\lambda_j^n = |a_j|$, $j = 1,\dots,N$. Using Lemma 20 we obtain

$$\sum_{j=1}^{N}|a_j| \le C + \sum_{j=1}^{N} a_j\lambda_j^n = C + \sum_{j=1}^{N} p_n(\lambda_j e_j) = \frac{1}{n^N}\sum_{l=1}^{n^N} p\left(\sum_{j=1}^{N} r_j^{n,N}(l)\lambda_j e_j\right),$$

where p_n denotes the n-homogeneous summand of p and $\{e_j\}_{j=1}^N$ is the canonical basis of $\mathbb{C}^N$. This finishes the proof of the finite-dimensional case. The infinite-dimensional case follows immediately.

□

The following result is an optimal generalisation of the Chebyshev theorem (Theorem **??**), which corresponds to $N = 1$.

Theorem 22 ([**MuSaTo:Complex**]). *Let $p \in \mathcal{P}^n(\mathbb{R}^N)$ and*

$$p(x_1,\dots,x_N) = \sum_{j=1}^{N} a_j x_j^n + \sum_{\substack{\beta\in\mathcal{J}(N,n)\\ \beta\neq(0,\dots,0,n,0,\dots,0)}} a_\beta x^\beta$$

Then

$$\sum_{j=1}^{N}|a_j| \le 2^{n-1}\max_{x_j\in[-1,1]}|p(x_1,\dots,x_N)|$$

and the constant 2^{n-1} is the best possible.

Proposition 23 ([**DimDin:SubspHolo**], [**DimGon:BlockPoly**]). *Let X be a Banach space over $\mathbb{K}$ with an unconditional Schauder basis and let $P \in \mathcal{P}(^nX)$ be a polynomial such that $|P(x_j)| \ge a > 0$, $j\in\mathbb{N}$, for some normalised block basic sequence $\{x_j\}$. Then $\mathcal{P}(^nX)$ contains ℓ_∞.*

PROOF. Let $\{e_j\}$ be the normalised unconditional Schauder basis of X. There exist sequences of integers $n_1 < m_1 < n_2 < m_2 < \cdots$ such that $\operatorname{supp} x_j \subset [n_j, m_j]$. Denote by $\pi_j\colon X\to X$ the projection $\pi_j\left(\sum_{l=1}^{\infty} a_l e_l\right) = \sum_{l=n_j}^{m_j} a_l e_l$ and define $P_j \in \mathcal{P}(^nX)$ by $P_j = P\circ\pi_j$.

Assume first that $\mathbb{K} = \mathbb{C}$. Fix $(a_j) \in B_{\ell_\infty}$ and $x \in X \setminus \{0\}$ and find $\zeta_j \in \mathbb{C}$ such that $\zeta_j^n = \frac{\overline{P_j(x)}}{|P_j(x)|}$, $j \in \mathbb{N}$. For any $k \in \mathbb{N}$ we obtain using Lemma 20

$$\sum_{j=1}^{k} |a_j P_j(x)| \le \sum_{j=1}^{k} |P_j(x)| = \sum_{j=1}^{k} P_j(\zeta_j x) = \frac{1}{n^k} \sum_{l=1}^{n^k} P\left(\sum_{j=1}^{k} r_j^{n,k}(l) \zeta_j \pi_j(x) \right)$$
$$\le \|P\| \max_{1 \le l \le n^k} \left\| \sum_{j=1}^{k} r_j^{n,k}(l) \zeta_j \pi_j(x) \right\|^n \le (2K)^n \|P\| \|x\|^n,$$

where K is the unconditional basis constant of $\{e_j\}$. It follows that the sum $\sum_{j=1}^{\infty} a_j P_j$ converges pointwise to a polynomial $Q \in \mathcal{P}(^n X)$ with $\|Q\| \le (2K)^n \|P\|$ (Theorem **??**).

On the other hand, choosing $m \in \mathbb{N}$ so that $|a_m| \ge \frac{1}{2} \|(a_j)\|_\infty$ and $\zeta \in \mathbb{K}$ with $\zeta^n = \frac{\overline{a_m}}{|a_m|}$, we have

$$\|Q\| = \sup_{x \in B_X} \left| \sum_{j=1}^{\infty} a_j P_j(x) \right| \ge \left| \sum_{j=1}^{\infty} a_j P_j(\zeta x_m) \right| = |a_m| |P(x_m)| \ge \frac{a}{2} \|(a_j)\|_\infty.$$

The real case follows immediately by passing to the complexification.

□

The next deeper results can be obtained by applying the theory of (p, q)-multiple summing operators. Given $m \in \mathbb{N}$, $1 \le p \le q \le \infty$, define

$$r(m, p, q) = \begin{cases} \frac{2m}{m + 2\left(\frac{1}{p} - \max\{\frac{1}{q}, \frac{1}{2}\}\right)} & \text{if } p < 2, \\ p & \text{if } p \ge 2. \end{cases}$$

Theorem 24 ([**DefSev:Littlewood**]). *For given $m \in \mathbb{N}$, $1 \le p \le q \le \infty$, and $r = r(m, p, q)$, there is $C_m > 0$ such that for every $A \in \mathcal{L}(^m c_0; \ell_p)$, in the complex setting, it holds*

$$\left(\sum_{i_1, \ldots, i_m} \|A(e_{i_1}, \ldots, e_{i_m})\|_q^r \right)^{\frac{1}{r}} \le C_m \|A\|.$$

Corollary 25 (Henri Frédéric Bohnenblust and Einar Hille, [**BohHil:DirichletSer**]). *If $B \in \mathcal{L}(^m c_0; \mathbb{C})$, then*

$$\left(\sum_{(i_1, \ldots, i_m) \in \mathbb{N}^m} |b_{i_1, \ldots, i_m}|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \le \left(\sqrt{2}\right)^{m-1} \|B\|.$$

For $m = 2$ this result is known as Litlewood's $\frac{4}{3}$-formula.

The polynomial case of Theorem 24 follows readily.

Theorem 26 ([**DefSev:Littlewood**]). *Let $\mathbb{K}$ be the scalar field. Given $m \in \mathbb{N}$, $1 \le p \le q \le \infty$, there is $D_m^{\mathbb{K}} > 0$ (depending on the scalar field) such that for every $P \in \mathcal{P}(^m c_0; \ell_p)$*

$$P(x) = \sum_{\alpha \in \mathcal{I}(\infty, d)} x^\alpha y_\alpha$$

$$\left(\sum_{i_1, \ldots, i_m} \|y_{i_1, \ldots, i_m}\|_q^r \right)^{\frac{1}{r}} \le D_m^{\mathbb{K}} \|P\|.$$

4. Tensor products and spaces of multilinear mappings

Given vector spaces $X_1, \dots, X_n$ over $\mathbb{K}$ we let Λ be the vector space of all formal linear combinations $\sum_{k=1}^{N} a_k (x_1^k \otimes \cdots \otimes x_n^k)$, $a_k \in \mathbb{K}$, $x_j^k \in X_j$. By Λ_0 we denote the linear subspace of Λ spanned by the vectors

$$a(x_1 \otimes \cdots \otimes x_n) - (x_1 \otimes \cdots \otimes (a x_k) \otimes \cdots \otimes x_n),$$
$$(x_1 \otimes \cdots \otimes (x_k + y_k) \otimes \cdots \otimes x_n) - (x_1 \otimes \cdots \otimes x_k \otimes \cdots \otimes x_n) - (x_1 \otimes \cdots \otimes y_k \otimes \cdots \otimes x_n),$$

where $k \in \{1, \dots, n\}$, $x_j, y_j \in X_j$, $a \in \mathbb{K}$. Then the quotient space Λ / Λ_0 is called the tensor product $X_1 \otimes \cdots \otimes X_n = \bigotimes_{j=1}^{n} X_j$. It is easily verified that tensor products are associative, so they can be build up inductively starting from the tensor product of a pair of vector spaces. Note also that by the definition of Λ_0 each $z \in X_1 \otimes \cdots \otimes X_n$ has a representation $z = \sum_{j=1}^{k} x_1^j \otimes \cdots \otimes x_n^j$. An element of $X_1 \otimes \cdots \otimes X_n$ that admits a representation $x_1 \otimes \cdots \otimes x_n$ is called an elementary tensor. The following is a fundamental observation: Given $\phi_j \in X_j^{\#}$, the function

$$\sum_{j=1}^{k} a_j (x_1^j \otimes \cdots \otimes x_n^j) \mapsto \sum_{j=1}^{k} a_j \phi_1(x_1^j) \cdots \phi_n(x_n^j) \tag{11}$$

is a linear form on the vector space Λ. Thus we obtain a useful criterion for distinguishing the vectors in a tensor product:

Proposition 27. *Let $X_1, \dots, X_n$ be vector spaces and $A_j \subset X_j^{\#}$ be subsets that separate the points of X_j, $j = 1, \dots, k$. Then $\sum_{j=1}^{k} a_j x_1^j \otimes \cdots \otimes x_n^j = 0$ in $X_1 \otimes \cdots \otimes X_n$ if and only if*

$$\sum_{j=1}^{k} a_j \phi_1(x_1^j) \cdots \phi_n(x_n^j) = 0$$

for every choice of $\phi_j \in A_j$.

Proof. $\Rightarrow$ The form (11) clearly vanishes on the generating vectors of Λ_0 and thus also on Λ_0.

$\Leftarrow$ By the definition of the space Λ_0 it is easily seen that we may assume that the vectors $\{x_l^j\}_{j=1}^k$ are linearly independent for each $l = 1, \dots, n$. Using an elementary linear algebra we obtain functionals $\{\phi_l^j\}_{j=1}^k \subset \operatorname{span} A_l$ biorthogonal to $\{x_l^j\}_{j=1}^k$. Hence $a_l = \sum_{j=1}^{k} a_j \phi_1^l(x_1^j) \cdots \phi_n^l(x_n^j) = 0$ for each $l = 1, \dots, k$.

□

Further, we define an n-linear mapping $\otimes \colon X_1 \times \cdots \times X_n \to \bigotimes_{j=1}^{n} X_j$ by $\otimes(x_1, \dots, x_n) = x_1 \otimes \cdots \otimes x_n$.

Theorem 28 (Universality of the tensor product – algebraic case)**.** *Let $X_1, \dots, X_n$, Y be vector spaces. For every n-linear mapping $M \in L(X_1, \dots, X_n; Y)$ there exists a unique linear operator $L_M \in L(X_1 \otimes \cdots \otimes X_n; Y)$ such that $M = L_M \circ \otimes$:*

$$\begin{tikzcd}
X_1 \times \cdots \times X_n \arrow[r, "M"] \arrow[d, "\otimes"'] & Y \\
X_1 \otimes \cdots \otimes X_n \arrow[ur, "L_M"'] &
\end{tikzcd}$$

The operator L_M satisfies

$$L_M(x_1 \otimes \cdots \otimes x_n) = M(x_1, \dots, x_n). \tag{12}$$

Proof. Using Proposition 27 it is easily seen that $\otimes$ maps products of linearly independent sets to linearly independent sets. Let A_j be the basis of X_j. We define L_M by the formula (12) on $\otimes(A_1 \times \cdots \times A_n)$ and extend it linearly onto the whole tensor product. Using the multilinearity it can be checked that the formula (12) still holds and it clearly uniquely determines L_M.

□

The operator L_M corresponding to M in the above theorem is called the linearisation of M. The following result is immediate.

Theorem 29. *Let $X_1, \dots, X_n$ be vector spaces. For $M \in L(X_1, \dots, X_n; \mathbb{K})$ and $z = \sum_{j=1}^{k} x_1^j \otimes \cdots \otimes x_n^j \in X_1 \otimes \cdots \otimes X_n$ put*

$$\langle M, z\rangle = \sum_{j=1}^{k} M(x_1^j, \dots, x_n^j) = \sum_{j=1}^{k} L_M(x_1^j \otimes \cdots \otimes x_n^j) = L_M(z).$$

Then $\langle L(X_1, \dots, X_n; \mathbb{K}), X_1 \otimes \cdots \otimes X_n\rangle$ forms a dual pair.

Tensor products of Banach spaces admit many non-equivalent natural norms. We will describe two important examples, namely the projective and the injective tensor norm. These norms can be shown to be in some sense extreme cases, as every "reasonable" tensor norm is bounded below by the injective norm and bounded above by the projective norm, [**Ryan:Tensor**].

Definition 30. Let $X_1, \dots, X_n$ be normed linear spaces. The projective tensor norm π on $X_1 \otimes \cdots \otimes X_n$ is defined by the formula

$$\pi(z) = \sup\{\langle M, z\rangle;\ M \in \mathcal{L}(X_1, \dots, X_n; \mathbb{K}), \|M\| \le 1\}, \quad z \in X_1 \otimes \cdots \otimes X_n.$$

The projective tensor product, denoted by $X_1 \otimes_\pi \cdots \otimes_\pi X_n$, is the completion of the normed linear space $(X_1 \otimes \cdots \otimes X_n, \pi)$.

A very useful alternative description of the projective tensor product and its norm is the following, see [**Ryan:Tensor**].

Proposition 31. *Let $X_1, \dots, X_n$ be normed linear spaces. Then for any $z \in X_1 \otimes_\pi \cdots \otimes_\pi X_n$ there exist bounded sequences $\{x_l^j\}_{j=1}^\infty \subset X_l$, $l = 1, \dots, n$, such that $z = \sum_{j=1}^\infty x_1^j \otimes \cdots \otimes x_n^j$ is an absolutely convergent series and*

$$\pi(z) = \inf\left\{\sum_{j=1}^{\infty} \|x_1^j\| \cdots \|x_n^j\|;\ z = \sum_{j=1}^{\infty} x_1^j \otimes \cdots \otimes x_n^j\right\}.$$

Furthermore, $\pi(x_1 \otimes \cdots \otimes x_n) = \|x_1\| \cdots \|x_n\|$ for every $x_j \in X_j$, $j = 1, \dots, n$.

This can be translated into a simple geometrical description of the unit ball of the projective tensor product:

$$B_{X_1 \otimes_\pi \cdots \otimes_\pi X_n} = \overline{\operatorname{conv}} \otimes (B_{X_1} \times \cdots \times B_{X_n}). \tag{13}$$

In particular, $\otimes\colon X_1 \times \cdots \times X_n \to X_1 \otimes_\pi \cdots \otimes_\pi X_n$ is a bounded n-linear mapping of norm 1. From the above it follows that the projective norm is defined so that the universality property of the tensor product remains valid also in the topological sense:

Theorem 32 (Universality of the tensor product – continuous case)**.** *Let $X_1, \dots, X_n$, Y be normed linear spaces. For every $M \in \mathcal{L}(X_1, \dots, X_n; Y)$ there exists a unique $L_M \in \mathcal{L}(X_1 \otimes_\pi \cdots \otimes_\pi X_n; Y)$ such that $M = L_M \circ \otimes$. The operator L_M satisfies* (12) *and the mapping $M \mapsto L_M$ is an isometry of the spaces $\mathcal{L}(X_1, \dots, X_n; Y)$ and $\mathcal{L}(X_1 \otimes_\pi \cdots \otimes_\pi X_n; Y)$.*

In particular, if $Y = \mathbb{K}$, then we get a simple but important duality relation.

Theorem 33. *Let $X_1, \dots, X_n$ be normed linear spaces. Then*

$$(X_1 \otimes_\pi \cdots \otimes_\pi X_n)^* = \mathcal{L}(X_1, \dots, X_n; \mathbb{K}),$$

where the evaluation is given by $\langle M, x_1 \otimes \cdots \otimes x_n\rangle = M(x_1, \dots, x_n)$.

If $n = 2$, then the canonical identification $\mathcal{L}(X_1, X_2; \mathbb{K}) = \mathcal{L}(X_1; X_2^*)$ of Fact **??** leads to an equivalent dual representation:

Fact 34. *Let X, Y be normed linear spaces. Then $(X \otimes_\pi Y)^* = \mathcal{L}(X; Y^*)$, where the evaluation is given by $\langle L, x \otimes y\rangle = L(x)(y)$.*

We continue by collecting some facts on symmetric tensor products and their close relationship with polynomials. Recall that the symmetrisation $\otimes_s\colon X \times \cdots \times X \to X \otimes \cdots \otimes X$ is a symmetric n-linear mapping given by

$$\otimes_s(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\eta \in S_n} \otimes(x_{\eta(1)}, \ldots, x_{\eta(n)}) = \frac{1}{n!} \sum_{\eta \in S_n} x_{\eta(1)} \otimes \cdots \otimes x_{\eta(n)},$$

where S_n is the set of all permutations of $\{1, \ldots, n\}$. We also use the notation $\otimes_s(x_1, \ldots, x_n) = x_1 \otimes_s \cdots \otimes_s x_n$ and $\otimes^n x = \otimes({}^n x) = x \otimes \cdots \otimes x$. The Polarisation formula (Proposition 6) yields that $\otimes_s^n X = \operatorname{span}\{\otimes^n x;\ x \in X\}$. The space $\otimes_s^n X$ is called a symmetric tensor product and the elements of $\otimes_s^n X$ are called symmetric tensors. When $\otimes_s^n X$ is equipped with the projective norm inherited from its superspace $\otimes_\pi^n X$, its completion becomes a closed subspace $\otimes_{\pi,s}^n X$ of $\otimes_\pi^n X$. Then the linearisation $\sigma_X^n\colon\ \otimes_\pi^n X \to \otimes_{\pi,s}^n X$ of $\otimes_s$ from Theorem 32 is a projection of norm 1. Thus we obtain the following:

Theorem 35 (Universality of the symmetric tensor product)**.** *Let X, Y be normed linear spaces. For every symmetric $M \in \mathcal{L}^s({}^n X; Y)$ there exists a unique $L_M \in \mathcal{L}(\otimes_{\pi,s}^n X; Y)$ such that $M = L_M \circ \otimes_s = L_M \circ \sigma_X^n \circ \otimes$. The mapping $M \mapsto L_M$ is an isometry of the spaces $\mathcal{L}^s({}^n X; Y)$ and $\mathcal{L}(\otimes_{\pi,s}^n X; Y)$. These facts are expressed by the following commutative diagram:*

$$\begin{tikzcd}
& X^n \arrow[dl, "\otimes"'] \arrow[d, "\otimes_s"] \arrow[dr, "M"] & \\
\otimes^n X \arrow[r, "\sigma_X^n"'] & \otimes_s^n X \arrow[r, "L_M"] & Y \\
& X \arrow[u, "\otimes^n"] \arrow[ur, "\widehat{M}"'] &
\end{tikzcd}$$

The following is immediate, using also Fact 34.

Corollary 36. *Let X, Y be normed linear spaces. Then the spaces $\mathcal{P}({}^n X; Y)$ and $\mathcal{L}(\otimes_{\pi,s}^n X; Y)$ are canonically isomorphic. In particular,*

$$(\otimes_{\pi,s}^n X)^* = \mathcal{P}({}^n X)$$

in the isomorphic sense, where the evaluation is given by $\langle P, \otimes^n x\rangle = P(x)$. More generally,

$$\big((\otimes_{\pi,s}^n X) \otimes_\pi Y\big)^* = \mathcal{L}(\otimes_{\pi,s}^n X; Y^*) = \mathcal{P}({}^n X; Y^*)$$

in the isomorphic sense, where the evaluation is given by $\langle P, \otimes^n x \otimes y\rangle = P(x)(y)$.

5. Weak continuity and spaces of polynomials

All topological spaces are assumed to be Hausdorff. Parallel to the concept of topology there is the notion of uniformity and of uniform spaces, see [**Engelking**] or [**Choq:Lectures**]. Since this concept is perhaps slightly less known we recall some of its basic properties. All the general results below can be found in [**Engelking**]. Let S be a non-empty set. Given a subset U of $S \times S$, denote $-U = \{(y, x);\ (x, y) \in U\}$, and if V is another subset of $S \times S$, let

$$U + V = \big\{(x, z);\ \text{there exists } y \in S \text{ such that } (x, y) \in U, (y, z) \in V\big\}.$$

A *uniformity* $\mathcal{U}$ in S is a filter consisting of subsets of $S \times S$ that satisfy the following:

(U1) $\Delta \subset U$ for all $U \in \mathcal{U}$, where $\Delta = \{(x, x);\ x \in S\}$;
(U2) if $U \in \mathcal{U}$, then $-U \in \mathcal{U}$;
(U3) for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V + V \subset U$;
(U4) $\bigcap_{U \in \mathcal{U}} U = \Delta$.

The pair $(S, \mathcal{U})$ is called a *uniform space*. Every uniform space $(S, \mathcal{U})$ becomes canonically a topological space with the basis of the topology consisting of all sets $U(x) = \{y \in S;\ (y, x) \in U\}$ for $x \in S$, $U \in \mathcal{U}$. The topology so defined is called the topology induced by the uniformity.

Every metric space (P, ρ) is a uniform space. Indeed, it suffices to consider the family $\mathcal{U}$ of all supersets of sets of the form $\{(x, y) \in P \times P;\ \rho(x, y) < 1/n\}$, $n \in \mathbb{N}$. Then $\mathcal{U}$ is a uniformity on P such that the induced topology is the topology defined by the metric. Every topological vector space (X, τ) has a unique translation-invariant uniformity defined on X that induces the topology τ. The uniformity $\mathcal{U}$ is defined as follows: $U \subset X \times X$ belongs to $\mathcal{U}$ if and only if $U = \{(x, y) \in X \times X;\ x - y \in V\}$, where V is a neighbourhood of 0. The case we are going to deal with mostly is the weak topology on a Banach space (X, w) or its restriction to some subset of X. We will call the canonical uniformity on (X, w) the weak uniformity on X (or on its subsets).

Topological spaces whose topology arises from a suitably chosen uniformity are called uniformisable. Uniformisable topological spaces are characterised as being Tychonov (i.e. completely regular) spaces. Equivalently, they are homeomorphic to subsets of $[0, 1]^{\Gamma}$. In general, the uniformity on a uniformisable space is not uniquely determined by the topology, but every compact topological space admits a unique uniformity.

A mapping f from a uniform space $(S, \mathcal{U})$ into another uniform space $(T, \mathcal{V})$ is called *uniformly continuous* if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ whenever $(x, y) \in U$. Obviously, every uniformly continuous mapping is continuous when S and T are endowed with their induced topologies. If $S \subset T$ is a subset of a uniform space $(T, \mathcal{U})$, then S is naturally a uniform space by using the restricted uniformity $\mathcal{V} = \{V;\ V = U \cap S \times S, U \in \mathcal{U}\}$. The identity mapping $Id\colon S \to T$ is then uniformly continuous. Note that any continuous linear mapping between topological vector spaces is w–w uniformly continuous.

A net $\{x_\gamma\}_{\gamma \in \Gamma}$ in a uniform space $(S, \mathcal{U})$ is said to be *Cauchy* if given $U \in \mathcal{U}$ there exists $\gamma_0 \in \Gamma$ such that $(x_\alpha, x_\beta) \in U$ for all $\alpha, \beta \in \Gamma$, $\alpha \succeq \gamma_0$, $\beta \succeq \gamma_0$. It is easy to see that a subnet of a Cauchy net is again a Cauchy net. Note that a net $\{x_\gamma\}_{\gamma \in \Gamma}$ in a topological vector space X is weakly Cauchy if and only if for each $f \in X^*$ the net $\{f(x_\gamma)\}_{\gamma \in \Gamma}$ is Cauchy (or equivalently convergent).

A uniform space $(S, \mathcal{U})$ is said to be *complete* if every Cauchy net in S converges. For every uniform space $(S, \mathcal{U})$ there exists a unique (up to a uniform isomorphism) completion $(\tilde{S}, \tilde{\mathcal{U}})$, i.e. a complete uniform space containing S as a dense subset. In particular, if $B \subset X$ is a bounded subset of a Banach space equipped with the uniformity inherited from (X, w), then its completion coincides with the w^*-closure of B in the bidual X^{**} equipped with the w^*-uniformity $(\overline{B}^{w^*}, w^*)$. A uniformly continuous mapping has a unique extension to a uniformly continuous mapping between the respective completions.

Let $(S, \mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. A subset A of S is said to be U-dense in S if for every $x \in S$ there is $y \in A$ such that $x \in U(y)$, i.e. $(x, y) \in U$. The space $(S, \mathcal{U})$ is called *totally bounded* if for every $U \in \mathcal{U}$ there exists a finite U-dense subset of S. The space $(S, \mathcal{U})$ is totally bounded if and only if every net in S has a Cauchy subnet ([**Choq:Lectures**]). A uniform space is compact if and only if it is totally bounded and complete. The completion of a totally bounded space is totally bounded and hence compact. A continuous mapping from a compact topological space (X, τ) is automatically uniformly continuous in the unique uniformity on X and the unique uniformity on its compact range.

A mapping between uniform spaces is called *Cauchy-continuous* (resp. *sequentially Cauchy-continuous*) if it maps Cauchy nets to Cauchy nets (resp. Cauchy sequences to Cauchy sequences). In particular, a function (or more generally a mapping into a complete space) is (sequentially) Cauchy-continuous if and only if it maps Cauchy nets (sequences) to convergent ones. It is easy to see that a uniformly continuous mapping is Cauchy-continuous (and hence also sequentially Cauchy-continuous). Further, a (sequentially) Cauchy-continuous mapping $f\colon (X, \mathcal{U}) \to (Y, \mathcal{V})$ is (sequentially) continuous. Indeed, given a net $\{x_\gamma\}_{\gamma \in \Gamma} \subset X$ converging to x, consider the directed set $\Lambda = \Gamma \times \{0, 1\}$ with the lexicographic ordering and set $x_{\gamma,0} = x_\gamma$ and $x_{\gamma,1} = x$. Then the net $\{x_\alpha\}_{\alpha \in \Lambda}$ is Cauchy and hence also $\{f(x_\alpha)\}_{\alpha \in \Lambda}$ is Cauchy. Now if $V \in \mathcal{V}$, then there is $\gamma_0 \in \Gamma$ such that $(f(x_{\gamma,0}), f(x_{\gamma,1})) \in V$ whenever $\gamma \succeq \gamma_0$, i.e. $f(x_\gamma) \in V(x)$. A Cauchy-continuous mapping from a uniform space X to a complete uniform space can be extended to a continuous mapping on the completion of X ([**Sche:HandAnal**]). The following observation is of great importance.

Proposition 37. *Let X be a totally bounded uniform space, Y a uniform space, and $f: X \to Y$ a Cauchy-continuous mapping. Then f is uniformly continuous.*

PROOF. The mapping f can be extended to a continuous mapping $F: \tilde{X} \to \tilde{Y}$, where $\tilde{X}$, $\tilde{Y}$ are the completions of X, Y. Since $\tilde{X}$ is compact, F is uniformly continuous, and thus so is f.

□

A closed, convex, and bounded subset of a normed linear space X will be abbreviated as a CCB set. Of great importance is the fact that a bounded set $B \subset X$ equipped with the uniformity inherited from (X, w) is totally bounded, which follows from the fact that $\overline{B}^{w^*} \subset X^{**}$ is w^*-compact. In particular, a mapping from B into a Banach space Y is weakly uniformly continuous if and only if it maps weakly Cauchy nets to convergent nets (Proposition 37).

We continue by giving a list of various notions of weak continuity that will be used in our investigations. Some of these classes have been introduced and studied by Richard Martin Aron and his co-authors, e.g. in [**ArHeVa:wCont**], [**AroPro:PolyAppr**], [**Aron:PolyAp**]; see also [**Dineen:Complex**].

Definition 38 ([**ArHeVa:wCont**])**.** Let X be a normed linear space, Y a Banach space, and $U \subset X$ a convex set. By $\mathcal{C}(U;Y)$ we denote the space $C(U;Y)$ endowed with the locally convex topology τ_b of uniform convergence on CCB subsets of U.

By $\mathcal{C}_{\mathrm{w}}(U;Y)$ we denote the linear subspace of $\mathcal{C}(U;Y)$ consisting of all mappings that are w–$\|\cdot\|$ continuous on CCB subsets of U.

By $\mathcal{C}_{\mathrm{wu}}(U;Y)$ we denote the linear subspace of $\mathcal{C}(U;Y)$ consisting of all mappings that are w–$\|\cdot\|$ uniformly continuous on CCB subsets of U. (In other words, $f \in \mathcal{C}_{\mathrm{wu}}(U;Y)$ if and only if for any CCB set V and any $\varepsilon > 0$ there are $\delta > 0$ and $\phi_1, \dots, \phi_k \in B_{X^*}$ such that $\|f(x) - f(y)\| < \varepsilon$ whenever $x, y \in V$ are such that $|\phi_j(x-y)| < \delta$ for $j = 1, \dots, k$.)

By $\mathcal{C}_{\mathrm{wsc}}(U;Y)$ we denote the linear subspace of $\mathcal{C}(U;Y)$ consisting of all mappings that are w–$\|\cdot\|$ sequentially continuous on CCB subsets of U, i.e. that map weakly convergent sequences in CCB subsets of U to convergent sequences in Y.

By $\mathcal{C}_{\mathrm{wsC}}(U;Y)$ we denote the linear subspace of $\mathcal{C}(U;Y)$ consisting of all mappings that are w–$\|\cdot\|$ sequentially Cauchy-continuous on CCB subsets of U, i.e. that map weakly Cauchy sequences in CCB subsets of U to convergent sequences in Y.

By $\mathcal{C}_{\mathrm{K}}(U;Y)$ we denote the linear subspace of $\mathcal{C}(U;Y)$ consisting of all mappings that map CCB subsets of U to relatively compact sets in Y.

By $\mathcal{C}_{\mathrm{wK}}(U;Y)$ we denote the linear subspace of $\mathcal{C}(U;Y)$ consisting of all mappings that map CCB subsets of U to relatively weakly compact sets in Y.

We use the usual convention for the notation when the range space is scalars, e.g. $\mathcal{C}_{\mathrm{wsc}}(U) = \mathcal{C}_{\mathrm{wsc}}(U;\mathbb{K})$.

We remark that if U is closed (in particular if $U = X$), then the topology on $\mathcal{C}(U;Y)$ is the topology of uniform convergence on bounded subsets of U; moreover, we may replace the CCB sets in the definitions above by bounded sets and $\mathcal{C}_{\mathrm{wsc}}(U;Y)$ (resp. $\mathcal{C}_{\mathrm{wsC}}(U;Y)$) are just mappings w–$\|\cdot\|$ sequentially continuous on U (resp. w–$\|\cdot\|$ sequentially Cauchy-continuous). Also, if X^* is separable, then it is easy to see that the weak uniformity on B_X is induced by the metric $\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x-y)|$, $\{f_n\}$ dense in B_{X^*}. Thus in this case $\mathcal{C}_{\mathrm{wsc}}(U;Y) = \mathcal{C}_{\mathrm{w}}(U;Y)$ and $\mathcal{C}_{\mathrm{wsC}}(U;Y) = \mathcal{C}_{\mathrm{wu}}(U;Y)$ (Proposition 37).

It is easy to prove that $\mathcal{C}_{\Psi}(U;Y)$, where Ψ is one of the properties w, wu, wsc, wsC, K, is a closed subspace of $\mathcal{C}(U;Y)$.

The following inclusions hold for any Banach space Y and any convex subset U of a normed linear space:

$$
\begin{array}{ccccc}
 & \rotatebox{45}{$\subset$} & \mathcal{C}_{\mathrm{K}}(U;Y) & \subset & \mathcal{C}_{\mathrm{wK}}(U;Y) \\
\mathcal{C}_{\mathrm{wu}}(U;Y) & \subset & \mathcal{C}_{\mathrm{w}}(U;Y) & \subset & \mathcal{C}_{\mathrm{wsc}}(U;Y) \\
 & \rotatebox{-45}{$\subset$} & \mathcal{C}_{\mathrm{wsC}}(U;Y) & \rotatebox{45}{$\subset$} &
\end{array}
$$

Corollary 39 ([**ArHeVa:wCont**]). *Let X be a normed linear space, Y a Banach space, and $n \in \mathbb{N}$. Then*

$$\begin{aligned} \mathcal{L}_{\mathrm{w}}({}^nX;Y) &= \mathcal{L}_{\mathrm{wu}}({}^nX;Y), \\ \mathcal{L}_{\mathrm{wsc}}({}^nX;Y) &= \mathcal{L}_{\mathrm{wsC}}({}^nX;Y), \\ \mathcal{P}_{\mathrm{w}}({}^nX;Y) &= \mathcal{P}_{\mathrm{wu}}({}^nX;Y), \\ \mathcal{P}_{\mathrm{wsc}}({}^nX;Y) &= \mathcal{P}_{\mathrm{wsC}}({}^nX;Y). \end{aligned}$$

From this and the relations shown earlier we obtain the following inclusions:

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\quad \subset \mathcal{P}_{\mathrm{K}}(X;Y) \subset \mathcal{P}_{\mathrm{wK}}(X;Y) \\ \mathcal{P}_{\mathrm{wu}}(X;Y) = \mathcal{P}_{\mathrm{w}}(X;Y) \subset \mathcal{P}_{\mathrm{wsc}}(X;Y) = \mathcal{P}_{\mathrm{wsC}}(X;Y) \end{array}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \subset \mathcal{L}_{\mathrm{wK}}(X;Y) \\ \mathcal{L}_{\mathrm{wu}}(X;Y) = \mathcal{L}_{\mathrm{K}}(X;Y) = \mathcal{L}_{\mathrm{w}}(X;Y) \subset \mathcal{L}_{\mathrm{wsc}}(X;Y) = \mathcal{L}_{\mathrm{wsC}}(X;Y) \end{array}$$

We remark that unlike the $\|\cdot\|$–$\|\cdot\|$ continuity, generally it is not sufficient to check the w–$\|\cdot\|$ continuity of polynomials only at the origin.

Example 40 ([**Aron:WUCEntire**]). Let $P \in \mathcal{P}({}^3\ell_2)$ be defined as $P(x) = x_1 \sum_{n=2}^{\infty} x_n^2$. Then the restriction of P to any bounded set is weakly continuous at the origin, but P is not weakly sequentially continuous. Indeed, $e_1 + e_n \overset{w}{\to} e_1$, but $P(e_1 + e_n) = 1$ and $P(e_1) = 0$.

The next two lemmata have numerous applications as they allow one to conclude, under suitable assumptions, that the origin is the point of weak discontinuity.

Lemma 41 ([**ChoHaLe:Extensions**]). *Let X be a normed linear space and Y a Banach space such that $\mathcal{P}({}^{n-1}X;Y) = \mathcal{P}_{\mathrm{wsc}}({}^{n-1}X;Y)$. If $P \in \mathcal{P}({}^nX;Y) \setminus \mathcal{P}_{\mathrm{wsc}}({}^nX;Y)$, then there is a weakly null sequence $\{y_k\}_{k=1}^{\infty} \subset S_X$ such that* $\inf_k \|P(y_k)\| > 0$.

PROOF. Suppose that $\{x_k\}_{k=1}^{\infty} \subset X$ is weakly convergent to $x \in X$ and $\|P(x_k) - P(x)\| \geq \varepsilon > 0$ for all $k \in \mathbb{N}$. Put $y_k = x_k - x$. By Lemma **??**

$$P(x_k) - P(x) = P(y_k) + \sum_{j=1}^{n-1} \binom{n}{j} \check{P}({}^jx, {}^{n-j}y_k)$$

for each $k \in \mathbb{N}$. By the assumption and Proposition **??** we have $\lim_{k\to\infty} \check{P}({}^jx, {}^{n-j}y_k) = 0$ for each $j \in \{1, \dots, n-1\}$. It follows that there is $k_0 \in \mathbb{N}$ such that $\|P(y_k)\| \geq \frac{\varepsilon}{2}$ for $k \geq k_0$. Finally, since $\liminf \|y_k\| > 0$ by the continuity of P, we can pass to the normalised sequence.

$\square$

Lemma 42 ([**DalHaj:Algebras**]). *Let X, Y be Banach spaces such that X does not contain ℓ_1 and $\mathcal{P}({}^{n-1}X;Y) = \mathcal{P}_{\mathrm{K}}({}^{n-1}X;Y)$. If $P \in \mathcal{P}({}^nX;Y) \setminus \mathcal{P}_{\mathrm{K}}({}^nX;Y)$, then there is a weakly null sequence $\{y_k\}_{k=1}^{\infty} \subset S_X$ such that $\{P(y_k)\}_{k=1}^{\infty}$ is not relatively compact.*

Theorem 43 ([**DalHaj:Algebras**]). *Let X be a normed linear space and $n \in \mathbb{N}$. Then $\mathcal{P}({}^nX;\ell_1) = \mathcal{P}_{\mathrm{K}}({}^nX;\ell_1)$ if and only if the space $\mathcal{P}({}^nX)$ does not contain c_0.*

PROOF. $\Rightarrow$ Assume that $\mathcal{P}({}^nX)$ contains c_0. By Corollary 36 the space $\mathcal{P}({}^nX)$ is isomorphic to $(\otimes_{\pi,s}^n X)^*$, and so $\otimes_{\pi,s}^n X$ contains a complemented subspace isomorphic to ℓ_1 by the Bessaga-Pełczyński theorem. Hence there is a $T \in \mathcal{L}(\otimes_{\pi,s}^n X;\ell_1)$ which is non-compact. Consequently $\widehat{T \circ \otimes_s} \notin \mathcal{P}_{\mathrm{K}}({}^nX;\ell_1)$ by Fact **??**.

$\Leftarrow$ If $\mathcal{P}({}^nX;\ell_1) \neq \mathcal{P}_{\mathrm{K}}({}^nX;\ell_1)$, then by Fact **??** there is a non-compact $T \in \mathcal{L}(\otimes_{\pi,s}^n X;\ell_1)$. Thus ℓ_1 is a quotient of $\otimes_{\pi,s}^n X$ by Proposition **??**. By the duality and Corollary 36 the space $\mathcal{P}({}^nX)$ then contains ℓ_∞, which is a contradiction.

$\square$

Definition 44. Let $1 \leq p, q \leq \infty$. We say that a sequence $\{x_j\}_{j=1}^{\infty}$ in a Banach space over $\mathbb{K}$ has an upper p-estimate (resp. lower q-estimate) if there exists $C > 0$ such that for every $n \in \mathbb{N}$ and every $a_1, \ldots, a_n \in \mathbb{K}$

$$\left\| \sum_{j=1}^{n} a_j x_j \right\| \leq C \left(\sum_{j=1}^{n} |a_j|^p \right)^{\frac{1}{p}}, \tag{14}$$

respectively

$$\left\| \sum_{j=1}^{n} a_j x_j \right\| \geq C \left(\sum_{j=1}^{n} |a_j|^q \right)^{\frac{1}{q}},$$

where the right-hand side is replaced by $\max_{j=1,\ldots,n} |a_j|$ if $p = \infty$ or $q = \infty$.

Fact 45. *Let X be a Banach space and $1 \leq p, q < \infty$. A sequence $\{x_j\}_{j=1}^{\infty} \subset X$ has an upper p-estimate if and only if the linear operator $T: \ell_p \to X$, $T(e_j) = x_j$, is bounded. A sequence $\{x_j\}_{j=1}^{\infty} \subset X$ has a lower q-estimate if and only if the linear operator $T: \overline{\operatorname{span}}\{x_j\} \to \ell_q$, $T(x_j) = e_j$, is bounded. In case $p = \infty$ we replace ℓ_p by c_0 and analogously for $q = \infty$.*

Definition 46 ([**KnaOde:Upperlp**]). Let $1 \leq p \leq \infty$. We say that a Banach space X has the S_p-property (resp. the T_p-property) if every normalised weakly null sequence has a subsequence with an upper p-estimate (resp. lower p-estimate).

The S_∞-property is equivalent to saying that every normalised weakly null sequence contains a subsequence equivalent to the basis of c_0.

We remark that the dual statement that property T_q of X implies property S_p of X^* does not hold: It is observed in [**GonJar:CompactPoly**] that a Banach space constructed in [**JohOde:SubsLplp**] is a counterexample.

Theorem 47 ([**Pelcz:Multilinear**], [**ArLaRyTo:Rademacher**], [**gj95b**]). *Let X, Y be Banach spaces and $P \in \mathcal{P}({}^nX; Y)$. If $n < p < +\infty$, then P takes sequences with an upper p-estimate into sequences with an upper $\frac{p}{n}$-estimate.*

Corollary 48 ([**GonJar:CompactPoly**]). *Let X be a Banach space with property S_p, $1 < p \leq \infty$. If $n < p$, then $\mathcal{P}^n(X) = \mathcal{P}^n_{\mathrm{wsC}}(X)$.*

PROOF. By Corollary 39 and Lemma 41 it suffices to show that $\lim_{k\to\infty} |P(x_k)| = 0$ for any weakly null sequence $\{x_k\}_{k=1}^{\infty}$ in X and any $P \in \mathcal{P}({}^nX)$, $1 \leq n < p$. By contradiction, suppose that $|P(x_k)| > \delta > 0$, $k \in \mathbb{N}$, for some weakly null sequence $\{x_k\}$. By passing to a subsequence we may assume that $\{x_k\}$ has an upper p-estimate. By Theorem 47 (resp. Theorem **??**) we conclude that $\{P(x_k)\}$ has an upper $\frac{p}{n}$-estimate. By considering the sums $\sum_{k=1}^{m} \frac{1}{t_k} t_k$ it is immediate that any sequence $\{t_k\}_{k=1}^{\infty}$ in $\mathbb{C}$ with an upper r-estimate, $r > 1$, is convergent to 0, which is a contradiction.

□

Since ℓ_p has property S_p and c_0 has property S_∞, we obtain the next result.

Corollary 49 ([**Bogd:wscc0**], [**Pelcz:Multilinear**]). *Let Γ be any set, $1 < p < \infty$, and $n \in \mathbb{N}$, $n < p$. Then*

$$\mathcal{P}^n(\ell_p) = \mathcal{P}^n_{\mathrm{wu}}(\ell_p),$$
$$\mathcal{P}(c_0) = \mathcal{P}_{\mathrm{wu}}(c_0).$$

On the other hand, if $n \geq p$, then $\sum_{j=1}^{\infty} x_j^n \in \mathcal{P}({}^n\ell_p) \setminus \mathcal{P}_{\mathrm{wsc}}({}^n\ell_p)$.

Theorem 50 (Raymond A. Ryan, [**Ryan:DunPet**]). *Let X be a normed linear space. The following statements are equivalent:*

(i) *X has the Dunford-Pettis property.*
(ii) *$\mathcal{L}_{\mathrm{wK}}(X; Y) \subset \mathcal{L}_{\mathrm{wsC}}(X; Y)$ for every Banach space Y.*

(iii) $\mathcal{L}_{\mathrm{wK}}({}^nX;Y) \subset \mathcal{L}_{\mathrm{wsC}}({}^nX;Y)$ *for every Banach space* Y *and every* $n \in \mathbb{N}$.
(iv) $\mathcal{P}_{\mathrm{wK}}(X;Y) \subset \mathcal{P}_{\mathrm{wsC}}(X;Y)$ *for every Banach space* Y.

We now continue developing our theory of weakly continuous mappings. The following result was shown for polynomials in [**ArHeVa:wCont**].

Theorem 51. *[***ChoHaLe:Extensions** *] Let* X, Y *be Banach spaces,* $\ell_1 \not\hookrightarrow X$, *and* $U \subset X$ *be a convex subset with non-empty interior. Then* $\mathcal{C}_w(U,Y) = \mathcal{C}_{\mathrm{wsc}}(U,Y)$.

PROOF. We only need prove that prove that $\mathcal{C}_{\mathrm{wsc}}(U,Y) \subset \mathcal{C}_{\mathrm{w}}(U,Y)$. By contradiction, let $T \in \mathcal{C}_{\mathrm{wsc}}(U,Y) \setminus \mathcal{C}_{\mathrm{w}}(U,Y)$ and suppose there is $\varepsilon > 0$, a CCB set $V \subset U$, and a bounded set $A \subset V$ such that $x \in w-cl(A)$, and $dist(T(x), T(A)) > \varepsilon$. By Kaplansky's theorem there exists a countable $S \subset A$, such that $x \in w-cl(S)$. Denote by Z a separable subspace of X containing $S \cup \{x\}$. As $\ell_1 \not\hookrightarrow Z$, by Theorem **??** there is a sequence $\{z_j\}_{j=1}^\infty$ in S weakly convergent to x. As $T \in \mathcal{C}_{\mathrm{wsc}}(U,Y)$, we have that $\lim_{j\to\infty} T(z_j) = T(x)$, a contradiction.

□

Theorem 52. *[***ChoHaLe:Extensions** *] Let* X, Y *be Banach spaces,* $\ell_1 \not\hookrightarrow X$, *and* $U \subset X$ *be a convex subset with non-empty interior. Then* $\mathcal{C}_{\mathrm{wu}}(U,Y) = \mathcal{C}_{\mathrm{wsC}}(U,Y)$.

PROOF. We only need prove that prove that $\mathcal{C}_{\mathrm{wsC}}(U,Y) \subset \mathcal{C}_{\mathrm{wu}}(U,Y)$. By contradiction, let $T \in \mathcal{C}_{\mathrm{wsC}}(U,Y)$ and $B \subset U$ be a CCB set such that T is not weakly uniformly continuous on B. Consider triples $\Phi = (P, A, \delta)$ of sets $P = \{\phi_1, \dots, \phi_n\} \subset B_{X^*}$, $A = \{a_1, \dots, a_n\} \subset \mathbb{R}$ and $\delta > 0$. For each Φ put

$$A_\Phi = \{x \in B;\ |\phi_j(x) - a_j| < \delta, j = 1, \dots, n\}.$$

$$B_\Phi = \{T(x);\ x \in A_\Phi\}.$$

Since T is not wu there exists an $\varepsilon > 0$ such that for every $P = \{\phi_1, \dots, \phi_n\} \subset B_{X^*}$, and $\delta > 0$, there exists $A = \{a_1, \dots, a_n\} \subset \mathbb{R}$ such that $\operatorname{diam} B_\Phi > \varepsilon$, where $\Phi = (P, A, \delta)$. The set of all Φ with this property will serve as an index set Γ. Put a partial order on the set Γ determined by the condition $\Phi_1 = (P_1, A_1, \delta_1) \preceq \Phi_2 = (P_2, A_2, \delta_2)$ if and only if

$$P_1 = \{\phi_1, \dots, \phi_n\} \subset P_2 = \{\phi_1, \dots, \phi_n, \phi_{n+1}, \dots, \phi_m\},$$

$$A_1 = \{a_1, \dots, a_n\} \subset A_2 = \{a_1, \dots, a_n, a_{n+1}, \dots, a_m\}, \delta_1 \geq \delta_2$$

Then $(\Gamma, \preceq)$ is a directed set. For each $\Phi \in \Gamma$, we choose $x_\Phi, y_\Phi \in A_\Phi$ such that $\|T(x_\Phi) - T(y_\Phi)\| > \varepsilon$. It is easy to see that the bounded net $\{x_\Phi - y_\Phi\}_{\Phi\in\Gamma} \subset X^{**}$ has a w^*-cluster point 0. By Kaplansky's theorem, there exists a countable subset $\{\Phi_n\}_{n=1}^\infty \subset \Gamma$, such that $\{x_{\Phi_n} - y_{\Phi_n}\}_{n\in\mathbb{N}}$ has a w^*-cluster point 0. By Theorem **??** the set $\{x_{\Phi_n} - y_{\Phi_n}\}_{n=1}^\infty$ contains a weakly null sequence $\{x_{\Phi_k} - y_{\Phi_k}\}_{k=1}^\infty$. By Rosenthal's ℓ_1-theorem, we may assume without loss of generality that both $\{x_{\Phi_k}\}_{k=1}^\infty$ and $\{y_{\Phi_k}\}_{k=1}^\infty$ are also weakly Cauchy. Since $\|T(x_{\Phi_k}) - T(y_{\Phi_k}\| > \varepsilon$, and the whole sequence $x_{\Phi_1}, y_{\Phi_1}, x_{\Phi_2}, y_{\Phi_2}, \dots$ is weakly Cauchy, we have reached a contradiction.

□

Definition 53. Let X, Y be normed linear spaces. We say that Y is crudely finitely representable in X if there is $K > 0$ such that for every finite-dimensional subspace F of Y there is a linear isomorphism $T\colon F \to T(F) \subset X$ satisfying $\|T\|\|T^{-1}\| \leq K$. We say that Y is finitely representable in X if for every $\varepsilon > 0$ the space Y is crudely finitely representable in X with the constant $K = 1 + \varepsilon$.

Observe that finite representability (and also crude) is a transitive relation. (Crude) finite representability clearly preserves properties that depend only on finite-dimensional subspaces (even in a uniform way independent of dimension). For example it preserves type or cotype, and moreover if Y is finitely representable in X, then $T_p(Y) \leq T_p(X)$ and $C_q(Y) \leq C_q(X)$. One of the key results in this area is the following principle of local reflexivity ([**LinRos:ScriptLp**], [**JoRoZi:Bases**]):

Theorem 54 (principle of local reflexivity, [**FHHMZ**])**.** *Let* X *be a Banach space,* $E \subset X^{**}$, $F \subset X^*$ *finite-dimensional subspaces, and* $\varepsilon > 0$. *Then there is a linear isomorphism* T *of* E *onto* $T(E) \subset X$ *such that* $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$, $f(T(\phi)) = \phi(f)$ *for any* $f \in F$ *and* $\phi \in E$, *and* T *is the identity on* $E \cap X$. *In particular,* X^{**} *is finitely representable in* X.

6. Ramsey theorem

Given a set X, we let $X^{(n)}$ to be the set of all subsets of X of cardinality n. We say that a system of k disjoint sets $\{S_i\}_{i=1}^k$ forms a *partitioning of* $X^{(n)}$ whenever $X^{(n)} = \bigcup_{i=1}^k S_i$.

Proposition 55. *Ramsey Let $k, n \in \mathbb{N}$. Then for every partitioning $\{S_i\}_{i=1}^k$ of $\mathbb{N}^{(n)}$ there exists $i \in \{1, \dots, k\}$ and an infinite set $M \subset \mathbb{N}$, such that $M^{(n)} \subset S_i$.*

An equivalent formulation is the following.

Let n be a natural number. Let ψ be a mapping from $\mathbb{N}^{(n)}$ to some finite set C. Then there is an infinite subset M of $\mathbb{N}$ such that ψ is constant on $M^{(n)}$.

Still a paraphrasis of the previous statement is the following: *If a coloring (with finite number of colors) of sets of natural numbers of a given length n is defined, then there is an infinite subset M of $\mathbb{N}$ such that all subsets of M of length n have the same color.*

PROOF. of Proposition 55 By induction on n. For $n = 1$ the result is obvious. Assume that for some $n > 1$ the statement has been proved for $1, 2, \dots, n-1$. We shall prove it for n.

The argument is based on the following observation: if we fix some $j \in \mathbb{N}$, we may consider all elements in $\mathbb{N}^{(n)}$ that contain j. Define a mapping ψ' from $(\mathbb{N} \setminus \{j\})^{(n-1)}$ into C as $\psi'(F) = \psi(F \cup \{j\})$ for all $F \in (\mathbb{N} \setminus \{j\})^{(n-1)}$. This is a coloring of all finite subsets of length $n-1$ in $(\mathbb{N} \setminus \{j\})^{(n-1)}$, hence, by the induction hypothesis, there exists an infinite subset M_1 of $\mathbb{N} \setminus \{j\}$ such that all subsets of M_1 of length $n-1$ get the same color (i.e., ψ' is constant on them). This means that ψ is constant (the same constant) on all sets $F \cup \{j\}$, where $F \in (\mathbb{N} \setminus \{j\})^{(n-1)}$.

To prove the assertion for n we iterate the construction above: let us start with $n_0 := 1$, and find an infinite subset M_1 of $\mathbb{N} \setminus \{n_0\}$ such that ψ is constant on all sets of the form $\{n_0\} \cup F$, for F a subset of length $n-1$ of M_1. Let $n_1 := \min M_1$ $(> n_0)$. Find an infinite subset M_2 of $M_1 \setminus \{n_1\}$ such that ψ is constant (maybe a different constant) on all sets of the form $\{n_1\} \cup F$, for F a subset of length $n-1$ of M_2, and put $n_2 := \min M_2$ $(> n_1)$. Continue in this way to obtain a sequence $M_1 \supset M_2 \supset \dots$ of infinite sets (and the sequence $n_0 < n_1 < n_2 < \dots$). By passing to a subsequence if necessary (denoted again $\{M_i\}$) we may assume that the same constant is associated to all M_i's. The sought set is then $\{n_i\}_{i=1}^\infty$.

□

By inspection of the proof we obtain the original finite version of the result.

Proposition 56. *Ramsey Let $k, n, m \in \mathbb{N}$. Then there exists $M = M(k, n, m)$ such that for every partitioning $\{S_i\}_{i=1}^k$ of $\{1, \dots, M\}^{(n)}$ there exists $i \in \{1, \dots, k\}$ and a subset $A \subset \{1, \dots, m\}$, $|A| = m$, such that $A^{(n)} \subset S_i$.*

Let $N \in \mathbb{N}$ and $e_1, \dots, e_N$ be the canonical basis of $\mathbb{R}^N$. We say that a polynomial $P \in \mathcal{P}(\mathbb{R}^N)$ is sub-symmetric if

$$P\left(\sum_{j=1}^k x_j e_{n_j}\right) = P\left(\sum_{j=1}^k x_j e_j\right)$$

whenever $1 \le k < N$, $x_1, \dots, x_k \in \mathbb{R}$, and $1 \le n_1 < \dots < n_k \le N$. Clearly, if $P \in \mathcal{P}(\mathbb{R}^N)$ is sub-symmetric and $1 \le n_1 < \dots < n_k \le N$, then the polynomial $Q \in \mathcal{P}(\mathbb{R}^k)$ given by $Q\left(\sum_{j=1}^k x_j e_j\right) = P\left(\sum_{j=1}^k x_j e_{n_j}\right)$ is also sub-symmetric.

Further, we set $\mathcal{N}(k, N) = \{\rho \in \mathcal{N}(k);\ \rho_k \le N\} = \{\rho \in \{1, \dots, N\}^k;\ \rho_1 < \dots < \rho_k\}$ and define the elementary sub-symmetric polynomials $P_\alpha^N \in \mathcal{P}({}^d\mathbb{R}^N)$ for $\alpha \in \mathcal{I}^+(k, d)$, $k \le N$, by

$$P_\alpha^N(x) = \sum_{\rho \in \mathcal{N}(k,N)} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k}. \tag{15}$$

We can see that the polynomials P_α^N, $\alpha \in \mathcal{I}^+(k, d)$, $k = 1, \dots, \min\{d, N\}$ form a basis of the space of sub-symmetric d-homogeneous polynomials on $\mathbb{R}^N$.

The concept of sub-symmetric polynomials on $\mathbb{R}^N$ can be used to capture the essential information on the behaviour of a given general polynomial.

Theorem 57. *Let $n, d \in \mathbb{N}$ and $\varepsilon > 0$. There exists an $N = N(n, d, \varepsilon)$ such that for every $P \in \mathcal{P}({}^d\ell_1^N)$, $\|P\| \le 1$, there exist $A \subset \{1, \dots, N\}$, $|A| = n$, and a sub-symmetric polynomial $Q \in \mathcal{P}({}^dY)$ such that $\|P \restriction_Y - Q\| < \varepsilon$, where $Y = \operatorname{span}\{e_k;\ k \in A\}$ and $\{e_k\}_{k=1}^N$ is the canonical basis of ℓ_1^N.*

PROOF. Given $P \in \mathcal{P}({}^d\ell_1^N)$ with $\|P\| \le 1$ there are $a_{\alpha,\rho} \in \mathbb{R}$ such that $P = \sum_{\alpha \in \mathcal{I}^+(d)} R_\alpha$, where

$$R_\alpha(x) = \sum_{\rho \in \mathcal{N}(k,N)} a_{\alpha,\rho} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k} \tag{16}$$

for $\alpha \in \mathcal{I}^+(k, d)$. By (**??**) and the Polarisation formula we see that each $|a_{\alpha,\rho}| < \binom{d}{\alpha}\|\check{P}\| \le d^d$. We show that for any $n \in \mathbb{N}$, $\varepsilon > 0$, $K > 0$, and $\alpha \in \mathcal{I}^+(d)$ there is $N = N_\alpha(n, \varepsilon, K)$ such that for any polynomial $R \in \mathcal{P}({}^d\ell_1^N)$ of the form (16) with $|a_{\alpha,\rho}| \le K$ for all $\rho \in \mathcal{N}(k, N)$ there is $A \subset \{1, \dots, N\}$, $|A| = n$, and $c \in \mathbb{R}$ such that $\|R \restriction_Y - cP_\alpha^n\| < \varepsilon$, where $Y = \operatorname{span}\{e_k;\ k \in A\}$. It is then clear that we may take

$$N(n, d, \varepsilon) = N_{\alpha^v}\big(\dots N_{\alpha^2}\big(N_{\alpha^1}(n, \tfrac{\varepsilon}{v}, d^d), \tfrac{\varepsilon}{v}, d^d\big) \dots, \tfrac{\varepsilon}{v}, d^d\big),$$

where $\alpha^1, \dots, \alpha^v$ is an enumeration of $\mathcal{I}^+(d)$.

So fix $\alpha \in \mathcal{I}^+(k, d)$, $n \in \mathbb{N}$, $\varepsilon > 0$, and $K > 0$. Let $\delta = \frac{\varepsilon}{2n!}$ and $M = \left[\frac{K}{\delta}\right]$. By Ramsey's theorem there is $N \in \mathbb{N}$ such that for every $2(M + 1)$-colouring of k-subsets (i.e. subsets of cardinality k) of $\{1, \dots, N\}$ there is $A \subset \{1, \dots, N\}$, $|A| = n$, such that all k-subset of A has the same colour. Now given $R \in \mathcal{P}({}^d\ell_1^N)$ of the form (16) with $|a_{\alpha,\rho}| \le K$ for all $\rho \in \mathcal{N}(k, N)$ we put $m(\rho) = \left[\frac{a_{\alpha,\rho}}{\delta}\right] \in \{-M-1, -M, \dots, M\}$. Note that $|a_{\alpha,\rho} - \delta m(\rho)| < \delta$. Each $\rho \in \mathcal{N}(k, N)$ uniquely determines a k-subset of $\{1, \dots, N\}$ and vice versa and so the function m induces a $2(M + 1)$-colouring of the k-subsets of $\{1, \dots, N\}$. Let $A \subset \{1, \dots, N\}$, $|A| = n$, be such that there is $m_0 \in \mathbb{N}$ satisfying $m(\rho) = m_0$ for all $\rho \subset A$. Then

$$\left| R\left(\sum_{j \in A} x_j e_j\right) - \delta m_0 P_\alpha^n\left(\sum_{j \in A} x_j e_j\right)\right| \le \delta \sum_{\rho \subset A} |x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k}| \le \delta \binom{n}{k} < \varepsilon$$

whenever $\left\|\sum_{j \in A} x_j e_j\right\| \le 1$.

□

7. Spreading models

In this section we develop some basic facts concerning the spreading model construction for a Banach space X, which leads to a Banach space with a sub-symmetric basis which captures the asymptotic behaviour of infinite sequences in X. We then pass to the closely related concepts of the Banach-Saks and the p-Banach-Saks properties. These results will be later applied to the weak sequential continuity properties of polynomials.

Definition 58. Let $K \ge 1$. We say that a sequence $\{x_n\}_{n=1}^\infty$ in a normed linear space is K-spreading if

$$\left\|\sum_{j=1}^k a_j x_{m_j}\right\| \le K \left\|\sum_{j=1}^k a_j x_{n_j}\right\|$$

whenever $k \in \mathbb{N}$, $a_1, \dots, a_k$ are any scalars, and $m_j, n_j \in \mathbb{N}$ are such that $m_1 < m_2 < \dots < m_k$, $n_1 < n_2 < \dots < n_k$.

In particular, a sequence $\{x_n\}$ is 1-spreading if and only if

$$\left\|\sum_{j=1}^k a_j x_{n_j}\right\| = \left\|\sum_{j=1}^k a_j x_j\right\|$$

whenever $k \in \mathbb{N}$, $a_1, \dots, a_k$ are any scalars, and $n_j \in \mathbb{N}$ are such that $n_1 < n_2 < \cdots < n_k$. Note that from Rosenthal's ℓ_1-theorem (Theorem **??**) it follows that any K-spreading sequence in a Banach space is either equivalent to the canonical basis of ℓ_1, or it is weakly Cauchy (use the fact that the linear operator $T\colon \operatorname{span}\{x_{n_j}\} \to \operatorname{span}\{x_j\}$, $T(x_{n_j}) = x_j$ is bounded and hence w–w uniformly continuous).

Proposition 59 ([**BeaLap:Modeles**]). *Let $\{e_n\}$ be a K-spreading sequence in a Banach space X. Then $\{e_n\}$ is a basic sequence if and only if it is not weakly convergent to a non-zero element of X. If moreover $\{e_n\}$ is weakly null, then $\{e_n\}$ is an unconditional basic sequence.*

Definition 60. A Schauder basis $\{e_n\}_{n=1}^\infty$ of a Banach space is called symmetric if $\{e_{\pi(n)}\}_{n=1}^\infty$ is equivalent to $\{e_n\}_{n=1}^\infty$ for any permutation π of $\mathbb{N}$. A Schauder basis $\{e_n\}_{n=1}^\infty$ of a Banach space is called sub-symmetric if it is unconditional and $\{e_{n_k}\}_{k=1}^\infty$ is equivalent to $\{e_n\}_{n=1}^\infty$ for every increasing sequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$.

We remark that a symmetric basis is automatically unconditional, and in fact sub-symmetric, [**Singer:Bases**]. It can be shown using the Uniform boundedness principle that a sub-symmetric basis is K-spreading for some $K \geq 1$ (the unconditionality here is substantial), and similarly if $\{e_n\}$ is a symmetric basis of a Banach space X, then there is $K \geq 1$ such that

$$\frac{1}{K}\left\|\sum_{n=1}^\infty a_n e_n\right\| \leq \left\|\sum_{n=1}^\infty a_n e_{\pi(n)}\right\| \leq K\left\|\sum_{n=1}^\infty a_n e_n\right\|$$

for every $\sum_{n=1}^\infty a_n e_n \in X$ and every permutation π of $\mathbb{N}$, [**Singer:Bases**]. Further, it is easy to check that if $\{e_n\} \subset X$ is a sub-symmetric basis that is K-spreading, then the sequence $\{f_n\} \subset X^*$ biorthogonal to $\{e_n\}$ is a sub-symmetric basic sequence that is $2CK$-spreading, where C is the unconditional basis constant of $\{e_n\}$.

Definition 61. Let $\{x_n\}$ be a sequence in a Banach space X. We say that a sequence $\{e_n\}$ in a Banach space Y is a spreading model of the sequence $\{x_n\}$ if for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that

$$(1-\varepsilon)\left\|\sum_{j=1}^k a_j e_j\right\| \leq \left\|\sum_{j=1}^k a_j x_{n_j}\right\| \leq (1+\varepsilon)\left\|\sum_{j=1}^k a_j e_j\right\|$$

for all $N \leq n_1 < n_2 < \cdots < n_k$ and all scalars $a_1, \dots, a_k$.

Theorem 62 (Antoine Brunel and Louis Sucheston, [**BruSuch:Bconv**]). *Let X be a Banach space and suppose that $\{x_n\} \subset X$ is a bounded sequence such that $\{x_n;\ n \in \mathbb{N}\}$ is not relatively compact. Then $\{x_n\}$ has a subsequence with a spreading model.*

The proof is based on repeated use of Ramsey's theorem and can be found e.g. in [**FHHMZ**] (the main idea of the crucial step is exposed in the proof of a simpler Theorem 57).

The following fact follows almost immediately from the definition.

Fact 63. *Let X, Y be Banach spaces and $\{x_n\} \subset X$ a sequence with a spreading model $\{e_n\} \subset Y$. Then $\overline{\operatorname{span}}\{e_n\}$ is finitely representable in X.*

The proof of the next proposition can be found in [**FHHMZ**].

Proposition 64. *Let X be a Banach space and $\{x_n\} \subset X$ a weakly null sequence with a spreading model $\{e_n\}$. Then $\{e_n\}$ is a sub-symmetric basic sequence with the unconditional basis constant at most* 2.

By passing to subsequences and diagonalising we obtain the following useful observation.

Proposition 65. *Let X, Y be a Banach spaces and $\{x_n\} \subset X$ a sequence with a spreading model $\{e_n\} \subset Y$. Let $\{\varepsilon_k\}_{k=1}^\infty \subset \mathbb{R}^+$ and $\{N_k\}_{k=1}^\infty \subset \mathbb{N}$. There is a subsequence $\{y_n\}$ of $\{x_n\}$ such that*

$$(1-\varepsilon_k)\left\|\sum_{j=1}^{N_k} a_j e_j\right\| \leq \left\|\sum_{j=1}^{N_k} a_j y_{n_j}\right\| \leq (1+\varepsilon_k)\left\|\sum_{j=1}^{N_k} a_j e_j\right\|, \tag{17}$$

for all $k \leq n_1 < n_2 < \cdots < n_{N_k}$ and all scalars $a_1, \dots, a_{N_k}$.

Later on we will make use of the following additional result. We prefer to omit the proof, as it can be obtained by modifying the proof of Theorem 62, working simultaneously with the norm $\|\cdot\|$ on X and P and keeping in mind that homogeneous polynomials form a closed set in the topology of uniform convergence on bounded sets (see also [**GonJar:Estimates**]).

Theorem 66. *Let X be a Banach space, $\{x_n\} \subset X$ a semi-normalised basic sequence, $P \in \mathcal{P}(^dX)$, $\{\varepsilon_k\}_{k=1}^{\infty} \subset \mathbb{R}^+$, and $\{N_k\}_{k=1}^{\infty} \subset \mathbb{N}$. There is a subsequence $\{y_n\}$ of $\{x_n\}$ with a spreading model $\{e_n\} \subset Y = \overline{\text{span}}\{e_n\}$ and a sub-symmetric polynomial $Q \in \mathcal{P}(^dY)$ such that (17) holds and*

$$\left| Q\left(\sum_{j=1}^{N_k} a_j e_j\right) - P\left(\sum_{j=1}^{N_k} a_j y_{n_j}\right)\right| \leq \varepsilon_k$$

for all $k \leq n_1 < n_2 < \cdots < n_{N_k}$ and all scalars $a_1, \dots, a_{N_k}$ with $\sum_{j=1}^{N_k} a_j y_{n_j} \in B_X$.

Definition 67. Let $p \in (1, \infty)$. A weakly null sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is said to be p-Banach-Saks sequence if for some $C > 0$,

$$\|\sum_{n=1}^{N} x_n\| \leq C N^{\frac{1}{p}}, \quad N \in \mathbb{N}. \tag{18}$$

We say that a weakly null sequence is hereditarily p-Banach-Saks sequence, if for some $C > 0$, every subsequence is p-Banach-Saks with the constant C. We say that X has the type p-Banach-Saks property if there exists a $C > 0$, such that every normalised weakly null sequence has a p-Banach-Saks subsequence with the constant C.

Lemma 68. *Let $p \in (1, \infty)$. Let $\{x_n\}_{n=1}^{\infty}$ be a normalised weakly null basic sequence in a Banach space X. The following statements are equivalent:*

1. *$\{x_n\}_{n=1}^{\infty}$ has a spreading model $\{e_i\}_{n=1}^{\infty}$ with a p-Banach-Saks basis,*
2. *$\{x_n\}_{n=1}^{\infty}$ has a hereditarily p-Banach-Saks subsequence $\{z_n\}_{n=1}^{\infty}$.*

PROOF. [**BeaLap:Modeles**] p. 45 Only $1 \Rightarrow 2$ requires a proof. Assume that $\{e_n\}_{n=1}^{\infty}$ is a hereditarily p-Banach-Saks sequence with the constant K. Let $\{z_n\}_{n=1}^{\infty}$ be a characteristic subsequence of $\{x_n\}$ from Definition **??** with the corresponding hereditarily p-Banach-Saks spreading basis $\{e_n\}_{n=1}^{\infty}$. We have for all $k \leq n_1 \leq \cdots \leq n_{2^k}$ and $a_1, \dots, a_{2^k} \in \mathbb{R}$,

$$\frac{1}{2}\|\sum_{j=1}^{2^k} a_j z_{n_j}\| \leq \|\sum_{j=1}^{2^k} a_j e_j\| \leq \frac{3}{2}\|\sum_{j=1}^{2^k} a_j z_{n_j}\|.$$

Fix $N \in \mathbb{N}, m = [\log_2(N)] + 1$. Note that $\lim_{N\to\infty} \frac{m}{N^{\frac{1}{2p}}} = 0$. For every subsequence $\{y_n\}_{n=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ we have

$$\frac{1}{N^{\frac{1}{p}}}\|\sum_{j=1}^{N} y_{n_j}\| \leq \frac{1}{N^{\frac{1}{p}}}\|\sum_{j=1}^{m} y_{n_j}\| + \frac{1}{N^{\frac{1}{p}}}\|\sum_{j=m+1}^{N} y_{n_j}\| \leq$$

$$\leq \frac{m}{N^{\frac{1}{p}}} + \frac{2}{N^{\frac{1}{p}}}\|\sum_{j=m+1}^{N} e_j\| \leq \frac{m}{N^{\frac{1}{2p}}} + \frac{2}{N^{\frac{1}{p}}}\|\sum_{j=1}^{N} e_j\| = C(N).$$

Because $\limsup_{N\to\infty} C(N) \leq 2K$, $\{z_n\}_{n=1}^{\infty}$ is a hereditarily p-Banach-Saks sequence with the constant $C = \sup_N C(N)$.

□

Lemma 69. *Let $\{x_i\}_{n=1}^{\infty}$ be a weakly null hereditarily p-Banach-Saks sequence in a Banach space X, $p > 1$. Then $\{x_i\}_{n=1}^{\infty}$ has an upper r-estimate for any $1 < r < p$.*

PROOF. Suppose $\frac{1}{p}+\frac{1}{q}=1$, $1<r<p$, and denote $D=\sum_{n=0}^{\infty}2^{nq(\frac{r}{p}-1)}<\infty$. Choose $\{a_j\}_{j=1}^{\infty}$ with $\sum_{j=1}^{\infty}|a_j|^r=1$. Define $B_n=\{j\in\mathbb{N};\ 2^{-n-1}<|a_j|\le 2^{-n}\}$, $n\in\mathbb{N}\cup\{0\}$. Then

$$\|\sum_{j=1}^{\infty}a_jx_j\|\le\sum_{n=0}^{\infty}\|\sum_{j\in B_n}a_jx_j\|\le C\sum_{n=0}^{\infty}2^{-n}|B_n|^{\frac{1}{p}}.$$

Using Holder's inequality

$$\sum_{n=0}^{\infty}2^{-n}|B_n|^{\frac{1}{p}}=\sum_{n=0}^{\infty}2^{n(\frac{r}{p}-1)}2^{-\frac{nr}{p}}|B_n|^{\frac{1}{p}}\le D^{\frac{1}{q}}(\sum_{n=0}^{\infty}2^{-nr}|B_n|)^{\frac{1}{p}}=$$

$$=2^{\frac{r}{p}}D(\sum_{n=0}^{\infty}2^{-(n+1)r}|B_n|)^{\frac{1}{p}}\le 2^{\frac{r}{p}}D(\sum_{j=1}^{\infty}|a_j|^r)^{\frac{1}{p}}=2^{\frac{r}{p}}D.$$

The last equality follows from the normalisation of $\{a_j\}_{j=1}^{\infty}$.

□

8. Polynomials and p-estimates

Definition 70. A sequence $\{x_k\}_{k=1}^{\infty}$ in a normed linear space is called $\mathcal{P}^n$-null if $\lim_{k\to\infty}P(x_k)=0$ for every $P\in\mathcal{P}^n(X)$, $P(0)=0$.

Lemma 71. *Let X be a real Banach space, $n\in\mathbb{N}$, $n\ge 2$, and let $\{x_k\}_{k=1}^{\infty}$ be a $\mathcal{P}^{n-1}$-null sequence in X. If $\lim_{k\to\infty}P(x_k)\neq 0$ for some $P\in\mathcal{P}(^nX)$, then $\{x_k\}$ has a subsequence with*

- *a lower n-estimate for non-negative coefficients,*
- *a lower r-estimate for every $r>n$,*
- *a lower n-estimate if n is even.*

PROOF. ([**Dev:VerySmooth**]) Without loss of generality we may suppose that $P(x_k)\ge 1$ for all $k\in\mathbb{N}$. Put $\delta_k=\frac{1}{2k^nn!2^k}$, $k\in\mathbb{N}$ and observe that for a fixed $m\in\mathbb{N}$ we have $\sum_{1\le k_1\le\cdots\le k_n=m}\delta_m\le\frac{1}{2n!2^m}$. Hence $\sum_{1\le k_1\le\cdots\le k_n}\delta_{k_n}\le\frac{1}{2n!}$. We will construct an increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that $|\check{P}(x_{k_1},\dots,x_{k_n})|<\delta_l$ whenever $\{k_j\}_{j=1}^{n}\subset\{n_k\}$, $k_1\le\cdots\le k_n$, $k_1<k_n$, and $k_n=n_l$. To this end, put $n_1=1$ and continue by induction. Suppose that $n_1,\dots,n_m$ are already defined for some $m\in\mathbb{N}$. For each $k_1,\dots,k_r\in\mathbb{N}$ the mapping $x\mapsto\check{P}(x_{k_1},\dots,x_{k_r},{}^{n-r}x)$ is a homogeneous polynomial of degree less than n. Thus there exists $n_{m+1}\in\mathbb{N}$, $n_{m+1}>n_m$ such that $|\check{P}(x_{k_1},\dots,x_{k_r},{}^{n-r}x_{n_{m+1}})|<\delta_{m+1}$ whenever $r<n$, $\{k_1,\dots,k_r\}\subset\{n_1,\dots,n_m\}$, and $k_1\le\cdots\le k_r$.

Denote $y_j=x_{n_j}$, $j\in\mathbb{N}$. Let $a_1,\dots,a_N\in\mathbb{R}$ and moreover $a_j\ge 0$, $j=1,\dots,N$ in case that n is odd. Then

$$\begin{aligned}P\left(\sum_{j=1}^{N}a_jy_j\right)&\ge\sum_{j=1}^{N}a_j^nP(y_j)-\sum_{\substack{1\le k_1\le\cdots\le k_n\le N\\ k_1<k_n}}n!\big|a_{k_1}\cdots a_{k_n}\check{P}(y_{k_1},\dots,y_{k_n})\big|\\&\ge\sum_{j=1}^{N}a_j^n-n!\sum_{\substack{1\le k_1\le\cdots\le k_n\le N\\ k_1<k_n}}|a_{k_1}\cdots a_{k_n}|\delta_{k_n}\ge\sum_{j=1}^{N}a_j^n-n!\sum_{\substack{1\le k_1\le\cdots\le k_n\le N\\ k_1<k_n}}\frac{a_{k_1}^n+\cdots+a_{k_n}^n}{n}\delta_{k_n}\\&\ge\sum_{j=1}^{N}a_j^n\left(1-n!\sum_{\substack{1\le k_1\le\cdots\le k_n\\ k_1<k_n}}\delta_{k_n}\right)\ge\frac{1}{2}\sum_{j=1}^{N}a_j^n.\end{aligned}$$

Thus

$$\left\|\sum_{j=1}^{N} a_j y_j\right\|^n \geq \frac{1}{\|P\|} P\left(\sum_{j=1}^{N} a_j y_j\right) \geq \frac{1}{2\|P\|} \sum_{j=1}^{N} a_j^n.$$

This proves the first and the third statement. The second statement follows from the first one and Corollary **??**. □

Proposition 72 (**[GonJar:Estimates]**). *Let X be a real Banach space that admits a $C^{k,\alpha}$-smooth bump, $k \in \mathbb{N}$, $\alpha \in (0,1]$. Then any normalised $\mathcal{P}^k$-null sequence in X has a subsequence with an upper $(k+\alpha)$-estimate.*

PROOF. Let $f\colon X \to \mathbb{R}$ be a $C^{k,\alpha}$-smooth function satisfying $f(0) = 0$ and $f(x) \geq 2$ for $\|x\| \geq 1$. For each $x \in X$ denote by $P(x) \in \mathcal{P}^k(X)$ the polynomial $h \mapsto T(x)[h] - f(x)$, where $T(x)$ is the Taylor polynomial of order k of f at x. Put $q = k + \alpha$. By Corollary **??** there is $K \geq 1$ such that $\left|f(x+h) - f(x) - P(x)[h]\right| \leq K\|h\|^q$ for all $h \in X$. Let $\{x_n\}_{n=1}^{\infty} \subset X$ be a normalised $\mathcal{P}^k$-null sequence in X. We will construct a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ by induction.

Suppose that $n_1 < n_2 < \cdots < n_{m-1} \in \mathbb{N}$ are already defined for some $m \in \mathbb{N}$. The mapping $x \mapsto P(x)$ is continuous and so the set

$$\mathcal{Q}_m = \left\{Q \in \mathcal{P}^k(X);\ Q(h) = P\left(\sum_{j=1}^{m-1} a_j x_{n_j}\right)[a_m h], |a_j| \leq 1, j = 1, \ldots, m\right\}$$

is a compact subset of $\mathcal{P}^k(X)$. Since $\{x_n\}$ is $\mathcal{P}^k$-null, it follows that there is $n_m \in \mathbb{N}$, $n_m > n_{m-1}$ such that $|Q(x_{n_m})| \leq \frac{1}{2^m}$ for all $Q \in \mathcal{Q}_m$.

Now if $a_1, \ldots, a_N \in \mathbb{R}$ are not all zero, we set $a = \left(\sum_{j=1}^{N} |a_j|^q\right)^{\frac{1}{q}}$ and $b_j = \frac{a_j}{aK^{1/q}}$. Then $\sum_{j=1}^{N} |b_j|^q = \frac{1}{K}$ and $|b_j| \leq 1$, $j = 1, \ldots, N$. Hence using the definition of $\mathcal{Q}_l$ we can estimate

$$\begin{aligned} f\left(\sum_{j=1}^{N} b_j x_{n_j}\right) &\leq \sum_{l=1}^{N} \left| f\left(\sum_{j=1}^{l} b_j x_{n_j}\right) - f\left(\sum_{j=1}^{l-1} b_j x_{n_j}\right)\right| \\ &\leq \sum_{l=1}^{N} K|b_l|^q + \left| P\left(\sum_{j=1}^{l-1} b_j x_{n_j}\right)[b_l x_{n_l}]\right| \leq \sum_{l=1}^{N} K|b_l|^q + \frac{1}{2^l} < 2, \end{aligned}$$

which implies that $\left\|\sum_{j=1}^{N} b_j x_{n_j}\right\| < 1$ and consequently $\left\|\sum_{j=1}^{N} a_j x_{n_j}\right\| < K^{\frac{1}{q}} \left(\sum_{j=1}^{N} |a_j|^q\right)^{\frac{1}{q}}$. □

9. Separating polynomials. Symmetric and sub-symmetric polynomials

Definition 73. Let X be a normed linear space with a Schauder basis $\{e_j\}_{j=1}^{\infty}$, and denote $X_0 = \operatorname{span}\{e_j\}_{j=1}^{\infty}$. For $\alpha \in \mathcal{I}^+(n,d)$ we define the polynomial $P_\alpha \in P({}^d X_0)$ by

$$P_\alpha(x) = \sum_{\rho \in \mathcal{N}(n)} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_n}^{\alpha_n}$$

for all $x = \sum x_j e_j \in X_0$. Clearly, each P_α is a sub-symmetric polynomial and it is called an elementary sub-symmetric polynomial. Further, we denote $s_d = P_{(d)}$, i.e.

$$s_d(x) = \sum_{j=1}^{\infty} x_j^d$$

for all $x = \sum x_j e_j \in X_0$. Clearly, each s_d is a symmetric polynomial and it is called a power sum symmetric polynomial.

In general the elementary sub-symmetric (or power sum symmetric) polynomials may not be bounded. They are, however, in the presence of a lower estimate.

Fact 74. *Let X be a Banach space with a Schauder basis $\{e_j\}_{j=1}^{\infty}$ that has a lower q-estimate, $1 \le q < \infty$, with a constant C. Let $\alpha \in \mathcal{I}^+(n,d)$ be such that $\alpha_l \ge q$, $l = 1, \dots, n$. Then*

$$|P_\alpha(x)| \le \prod_{l=1}^{n}\left(\sum_{j=1}^{k}|x_j|^{\alpha_l}\right) \le \left(\sum_{j=1}^{k}|x_j|^q\right)^{\frac{d}{q}} \le \frac{1}{C^d}\|x\|^d$$

for every finitely supported $x \in X$, $x = \sum_{j=1}^{k} x_j e_j$. In particular, P_α can be uniquely extended to a continuous d-homogeneous polynomial on the whole of X, which is then sub-symmetric in case that $\{e_j\}$ is K-spreading.

Similarly, each s_d, $d \ge q$, can be uniquely extended to a continuous d-homogeneous polynomial on the whole of X, which is then symmetric in case that $\{e_j\}$ is symmetric. Conversely, if $\{e_j\}$ is unconditional and s_d is bounded on span$\{e_j\}$*, then $\{e_j\}$ has a lower d-estimate.*

PROOF. The estimates are clear using $\sum_{j=1}^{k}|x_j|^{\alpha_l} \le \left(\sum_{j=1}^{k}|x_j|^q\right)^{\alpha_l/q}$. The extension follows from Proposition **??**. Considering the assumption of the unconditional basis in the last statement, check s_1 on c_0 with the summing basis.

$\square$

For a sub-symmetric homogeneous polynomial the coefficients $y_{\alpha,\rho}$ in (**??**) do not depend on ρ by (**??**) and the Polarisation formula. Thus we obtain the following.

Fact 75. *Let X be a normed linear space with a Schauder basis $\{e_j\}_{j=1}^{\infty}$, Y a vector space, and $X_0 =$* span$\{e_j\}$*. If $P \in P(^dX_0; Y)$ is sub-symmetric, then there is a unique collection of vectors $\{y_\alpha;\ \alpha \in \mathcal{I}^+(d)\} \subset Y$ such that for every $x \in X_0$*

$$P(x) = \sum_{\alpha \in \mathcal{I}^+(d)} P_\alpha(x) y_\alpha. \tag{19}$$

The coefficients y_α are given by $y_\alpha = \binom{d}{\alpha}\check{P}(^{\alpha_1}e_1, \dots, {}^{\alpha_n}e_n)$ when $\alpha \in \mathcal{I}^+(n,d)$.

The usefulness of sub-symmetric polynomials stems from the fact that the space of sub-symmetric polynomials is finite-dimensional (the previous fact), while a general polynomial can be asymptotically approximated by a sub-symmetric one (Theorem 66).

The next theorem was proved in an unpublished version of [**HabHaj:StabPoly**] for ℓ_p spaces. The averaging proof of the slightly more general formulation here is due to [**Gon:Spread**].

Theorem 76. *Let X be a Banach space with a sub-symmetric Schauder basis $\{e_j\}_{j=1}^{\infty}$ and let $P \in \mathcal{P}(^dX)$ be a sub-symmetric polynomial. If P satisfies* (19) *and $q = \min\{\alpha_j;\ \alpha \in \mathcal{I}^+(d), y_\alpha \ne 0\}$, then $\{e_j\}$ has a lower q-estimate.*

PROOF. The proof is similar to the proof of Lemma **??**. Since the basis is sub-symmetric, it is automatically semi-normalised and in particular it is bounded by $M \ge 1$. It is easy to check that if X is real, then $\{e_j\}$ is also a sub-symmetric basis of $\tilde{X}$ and the extension $\tilde{P}$ is sub-symmetric. Thus we may without loss of generality assume that X is complex. Further, we assume for simplicity that there is $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{I}^+(d)$ such that $y_\alpha \ne 0$ and $\alpha_1 = q$. Put $b = \check{P}(^{\alpha_1}e_1, \dots, {}^{\alpha_k}e_k)$. By Fact 75, $y_\alpha = \binom{d}{\alpha}b$ and hence $b \ne 0$. Since P is sub-symmetric, $\check{P}(^{\alpha_1}e_{n_1}, \dots, {}^{\alpha_k}e_{n_k}) = b$ for any $n_1, \dots, n_k \in \mathbb{N}$, $n_1 < \dots < n_k$.

Now fix any $x_1,\dots,x_N \in \mathbb{C}$ and choose $\xi_1,\dots,\xi_N \in \mathbb{C}$, $|\xi_j| = 1$, such that $\xi_j^q x_j^q b = |x_j^q b|$. Then using Lemma 20 we obtain

$$|b|\sum_{j=1}^{N}|x_j|^q = \sum_{j=1}^{N}\xi_j^q x_j^q \check{P}({}^{\alpha_1}e_j, {}^{\alpha_2}e_{N+2},\dots,{}^{\alpha_k}e_{N+k}) = \sum_{j=1}^{N}\check{P}\big({}^{q}(\xi_j x_j e_j), {}^{\alpha_2}e_{N+2},\dots,{}^{\alpha_k}e_{N+k}\big)$$

$$= \frac{1}{q^N}\sum_{l=1}^{q^N}\check{P}\left({}^{q}\left(\sum_{j=1}^{N} r_j^{q,N}(l)\xi_j x_j e_j\right), {}^{\alpha_2}e_{N+2},\dots,{}^{\alpha_k}e_{N+k}\right) \le 2CM^d\|\check{P}\|\left\|\sum_{j=1}^{N}x_je_j\right\|^q,$$

where C is the unconditional basis constant of $\{e_j\}$.

□

Corollary 77. *Let $P \in \mathcal{P}({}^d\ell_p)$, $1 \le p < \infty$ be a sub-symmetric polynomial. If P satisfies* (19)*, then $\alpha_j \ge p$, $j = 1,\dots,n$ for every $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathcal{I}^+(d)$ with $y_\alpha \ne 0$.*

Proposition 78 ([**GoGoJa:SymPoly**]). *Let X be a Banach space with a K-spreading (resp. symmetric) basis $\{e_j\}_{j=1}^\infty$. If X admits a k-homogeneous separating polynomial, then there exists a k-homogeneous sub-symmetric (resp. symmetric) separating polynomial on X.*

Corollary 79 ([**Kurz:Approx**], [**GonJar:Estimates**]). *Let X be a Banach space with a sub-symmetric basis $\{e_j\}_{j=1}^\infty$ and a separating polynomial. Then $\{e_j\}$ is equivalent to the canonical basis of some ℓ_p, p an even integer.*

PROOF. By Fact **??** and Proposition 78 we may assume that there is a sub-symmetric $P \in \mathcal{P}({}^dX)$ with $P(x) \ge 1$ whenever $\|x\| = 1$. Let P be given by (19). By Theorem 76 there is $p \in \mathbb{N}$ such that $\{e_j\}$ has a lower p-estimate with a constant C. By Fact 74 we have

$$\|x\|^d \le P(x) = \sum_{\alpha\in\mathcal{I}^+(d)} P_\alpha(x)y_\alpha \le \sum_{\alpha\in\mathcal{I}^+(d)}|y_\alpha|\left(\sum_{j=1}^{k}|x_j|^p\right)^{\frac{d}{p}} \le \frac{1}{C^d}\|x\|^d\sum_{\alpha\in\mathcal{I}^+(d)}|y_\alpha|$$

for any $x = \sum_{j=1}^k x_je_j \in X$. Finally, p must be even by Corollary **??**.

□

Theorem 80 ([**GonGon:SepPoly**]). *Let X be a Banach space with a separating d-homogeneous polynomial. Then every normalised weakly null sequence $\{x_j\}_{j=1}^\infty$ in X has a subsequence equivalent to the canonical basis of ℓ_p, p even integer, such that $p \le q(X)$ and d is an integer multiple of p.*

PROOF. Let n be the greatest integer such that $\{x_j\}_{j=1}^\infty$ is $\mathcal{P}^{n-1}$-null. Clearly $n \le d$. By Proposition 72 we can assume that $\{x_j\}_{j=1}^\infty$ has an upper n-estimate. Passing to further subsequence using Theorem 62 we may assume that $\{x_j\}_{j=1}^\infty$ has a spreading model $\{e_j\}_{j=1}^\infty$ that is sub-symmetric by Proposition 64. Let $Y = \overline{\operatorname{span}}\{e_j\}$. By Corollary **??** and Fact 63, Y admits a d-homogeneous separating polynomial. By Corollaries 79 and **??**, $\{e_j\}_{j=1}^\infty$ is equivalent to the canonical basis of ℓ_p, where p is an even integer and d is an integer multiple of p. The upper n-estimate clearly passes to $\{e_j\}_{j=1}^\infty$, so we have $n \le p$.

By Lemma 71, $\{x_j\}_{j=1}^\infty$ has a subsequence with a lower r-estimate for every $r > n$. Hence $\{e_j\}_{j=1}^\infty$ has a lower r-estimate for every $r > n$, so $n \ge p$. It follows that $n = p$, which is an even integer. Thus by Lemma 71 a subsequence of $\{x_j\}_{j=1}^\infty$ has a lower p-estimate. Consequently, this subsequence is equivalent to the canonical basis of ℓ_p.

The fact that $q(\ell_p) = p \le q(X)$ follows automatically, as the cotype passes to subspaces.

□

The following result contains the main structural information concerning C^∞-smooth Banach spaces. It is due to Robert Deville (except for the equality $d = mq$).

Theorem 81 ([**Dev:VerySmooth**]). *Let X be a Banach space which does not contain a subspace isomorphic to c_0 and admits a C^k-smooth bump for every $k \in \mathbb{N}$. Then X has a d-homogeneous separating polynomial.*

Moreover, X has an exact cotype $q = q(X)$ which is an even integer such that $d = mq$ for some $m \in \mathbb{N}$, and X has a subspace isomorphic to ℓ_q

10. Stabilisation of polynomials

Note that every spreading model of $\{e_j\}$ is again equal to $\{e_j\}$ (in the isometric sense). Using Theorem 66 and passing finitely many times to subsequences, for every $P \in \mathcal{P}^d(\ell_p)$ there exist an increasing sequence $M = \{m_j\} \subset \mathbb{N}$ and a sub-symmetric polynomial $Q \in \mathcal{P}^d(\ell_p)$ such that for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that

$$\left| Q\left(\sum_{j=1}^{k} x_j e_j\right) - P\left(\sum_{j=1}^{k} x_j e_{n_j}\right)\right| \leq \varepsilon \tag{20}$$

for all $N \leq n_1 < n_2 < \cdots < n_k$, $n_j \in M$ and all $\sum_{j=1}^{k} x_j e_{n_j} \in B_{\ell_p}$. We show that in fact a substantially stronger result holds:

Theorem 82 ([**HabHaj:StabPoly**]). *Let $P \in \mathcal{P}^d(\ell_p)$, $1 \leq p < \infty$. There are a subsequence $\{e_{n_j}\}$ of the canonical basis $\{e_j\}$ and a sub-symmetric polynomial $Q \in \mathcal{P}^d(\ell_p)$ such that for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that*

$$\left| Q\left(\sum_{j=N}^{\infty} x_j e_j\right) - P\left(\sum_{j=N}^{\infty} x_j e_{n_j}\right)\right| \leq \varepsilon$$

for every $\sum_{j=N}^{\infty} x_j e_{n_j} \in B_{\ell_p}$. In other words,

$$\lim_{N\to\infty} \left\| (Q - P)\restriction_{\overline{\operatorname{span}}\{e_{n_j}\}_{j=N}^{\infty}} \right\| = 0.$$

We continue by applying these results in the setting of general Banach spaces.

Definition 83. Let X be a real Banach space with a Schauder basis $\{e_j\}_{j=1}^{\infty}$ and let $f\colon X \to \mathbb{R}$ be a p-homogeneous function (i.e. $f(\lambda x) = \lambda^p f(x)$ for all $\lambda > 0$). We say that f stabilises at zero level if

$$\lim_{n\to\infty} \sup\left\{|f(x)|;\ x = \sum_{j=n}^{\infty} a_j e_j, x \in B_X\right\} = 0.$$

It would be more precise to say that f stabilises at zero level with respect to the basis $\{e_j\}$. Note that if f stabilises at zero level with respect to the basis $\{e_j\}$ and $\{u_j\}$ is a block basis of $\{e_j\}$, then $f\restriction_{\overline{\operatorname{span}}\{u_j\}}$ stabilises at zero level with respect to the basis $\{u_j\}$.

Theorem 84 ([**HabHaj:StabPoly**]). *Let X be a real infinite-dimensional Banach space and $P \in \mathcal{P}(^dX)$. Then there exists a basic sequence $\{x_j\}_{j=1}^{\infty} \subset X$ such that $P\restriction_{\overline{\operatorname{span}}\{x_j\}}$ is either separating or stabilises at zero level.*

11. Sub-symmetric polynomials on $\mathbb{R}^n$

The main result of this section (Corollary 91) concerns the properties of sub-symmetric polynomials, however in its proof we need to work also with partial derivatives of the polynomials P_α^N and for this reason we consider also the polynomials P_α^N given by the formula (15) where $\alpha \in \mathcal{I}(k,d)$, $k \leq N$. We denote by $H^{d,K}(\mathbb{R}^N)$ the subspace of $\mathcal{P}^d(\mathbb{R}^N)$ generated by the polynomials P_α^N, $\alpha \in \bigcup_{k=1}^{K} \mathcal{J}(k,d)$. For formal reasons we also put $P_\alpha^N = 0$ if $k > N$ and $P_{()}^N = 1$, both even for $N = 0$, further $\mathcal{I}(0,0) = \{()\}$, and $\mathbb{R}^0 = \{0\}$. Note that these definitions are consistent with (15).

The following fact describes an important relation between the restriction of P_α^M to the first N coordinates and P_α^N. Note that for $M > N$ we consider canonically $\mathbb{R}^N$ as a subspace of $\mathbb{R}^M$.

Fact 85. *Let $M, N, k, d \in \mathbb{N}_0$, $N < M$, and $\alpha \in \mathcal{I}(k,d)$ be such that $\alpha_m > 0$ and $\alpha_{m+1} = \cdots = \alpha_k = 0$ for some $0 \le m \le k$. Then*

$$P_\alpha^M(x) = \sum_{j=m}^{k} \binom{M-N}{k-j} P^N_{(\alpha_1,\ldots,\alpha_j)}(x)$$

for every $x \in \mathbb{R}^N$. Conversely,

$$P_\alpha^N(x) = \sum_{j=m}^{k} (-1)^{k-j} \binom{M-N+k-j-1}{k-j} P^M_{(\alpha_1,\ldots,\alpha_j)}(x)$$

for every $x \in \mathbb{R}^N$.

PROOF. The first relation follows from the following (recall that $x \in \mathbb{R}^N$, i.e. $x_{N+1} = \cdots = x_M = 0$ as per the aforementioned convention):

$$\begin{aligned} P_\alpha^M(x) &= \sum_{\rho \in \mathcal{N}(k,M)} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_m}^{\alpha_m} = \sum_{\substack{\rho \in \mathcal{N}(k,M) \\ \rho_m \le N}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_m}^{\alpha_m} \\ &= \sum_{j=m}^{k} \sum_{\substack{1 \le \rho_1 < \cdots < \rho_k \le M \\ \rho_j \le N < \rho_{j+1}}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_m}^{\alpha_m} = \sum_{j=m}^{k} \binom{M-N}{k-j} P^N_{(\alpha_1,\ldots,\alpha_j)}(x). \end{aligned}$$

The second relation can be proved by induction on $k-m$. For $k-m=0$ it follows immediately from the first one. For the induction step we use the first relation together with the inductive hypothesis to obtain

$$\begin{aligned} P_\alpha^N(x) &= P_\alpha^M(x) - \sum_{j=m}^{k-1} \binom{M-N}{k-j} P^N_{(\alpha_1,\ldots,\alpha_j)}(x) \\ &= P_\alpha^M(x) - \sum_{j=m}^{k-1} \binom{M-N}{k-j} \sum_{l=m}^{j} (-1)^{j-l} \binom{M-N+j-l-1}{j-l} P^M_{(\alpha_1,\ldots,\alpha_l)}(x) \\ &= P_\alpha^M(x) - \sum_{l=m}^{k-1} \left(\sum_{j=l}^{k-1} (-1)^{j-l} \binom{M-N}{k-j} \binom{M-N+j-l-1}{j-l} \right) P^M_{(\alpha_1,\ldots,\alpha_l)}(x) \end{aligned}$$

and the result now follows from the identity $\sum_{j=l}^{k} (-1)^{j-l} \binom{M-N}{k-j} \binom{M-N+j-l-1}{j-l} = 0$. Adding or removing a couple of zero summands, this is equivalent to $\sum_{p=0}^{M-N} (-1)^{k-l-p} \binom{M-N}{p} \binom{M-N+k-l-p-1}{M-N-1} = 0$, which is the Fréchet formula for the polynomial $t \mapsto \binom{M-N+k-l-t-1}{M-N-1}$ of degree $M-N-1$ (Theorem **??**). ∎

It is very important to notice that the previous fact covers all the special cases like $N < k \le M$, $k > M$, $N = 0$, $m = 0$, or $k = 0$. Observe also that in particular in the sub-symmetric case (i.e. $\alpha \in \mathcal{I}^+(d)$) we have $P_\alpha^M \restriction_{\mathbb{R}^N} = P_\alpha^N$. Hence for the elementary sub-symmetric polynomials the superscript N can be dropped. We will use this simplification for the power sum symmetric polynomials $s_n^N = P_{(n)}^N = s_n$.

The next fact deals with the situation when we fix the first N coordinates of P_α^M.

Fact 86. *Let $N, d \in \mathbb{N}_0$, $M, k \in \mathbb{N}$, $N < M$, $k \le M$, $\alpha \in \mathcal{I}(k,d)$, and $y \in \mathbb{R}^N$. Then the polynomial $(x_1, \ldots, x_{M-N}) \mapsto P_\alpha^M(y_1, \ldots, y_N, x_1, \ldots, x_{M-N})$ belongs to $H^{d,\min\{k,M-N\}}(\mathbb{R}^{M-N})$.*

PROOF.

$$P_\alpha^M(y_1,\dots,y_N,x_1,\dots,x_{M-N}) = \sum_{j=0}^{k} \sum_{\substack{1\le\rho_1<\dots<\rho_k\le M\\ \rho_j\le N<\rho_{j+1}}} y_{\rho_1}^{\alpha_1}\cdots y_{\rho_j}^{\alpha_j} x_{\rho_{j+1}-N}^{\alpha_{j+1}}\cdots x_{\rho_k-N}^{\alpha_k}$$
$$= \sum_{\substack{0\le j\le k\\ k-(M-N)\le j\le N}} P^N_{(\alpha_1,\dots,\alpha_j)}(y) P^{M-N}_{(\alpha_{j+1},\dots,\alpha_k)}(x_1,\dots,x_{M-N}).$$

□

Let $k,d\in\mathbb{N}$, $\alpha\in\mathcal{I}(k,d)$, $k\le N$, $x\in\mathbb{R}^N$, and $1\le l\le N$. Then

$$\frac{\partial P_\alpha^N}{\partial x_l}(x) = \frac{\partial}{\partial x_l}\left(\sum_{j=1}^{k} \sum_{\substack{\rho\in\mathcal{N}(k,N)\\ \rho_j=l}} x_{\rho_1}^{\alpha_1}\cdots x_{\rho_k}^{\alpha_k}\right) = \sum_{\substack{j=1\\ \alpha_j>0}}^{k} \alpha_j \sum_{\substack{1\le\rho_1<\dots<\rho_{j-1}<l\\ l<\rho_{j+1}<\dots<\rho_k\le N}} x_{\rho_1}^{\alpha_1}\cdots x_{\rho_{j-1}}^{\alpha_{j-1}} x_l^{\alpha_j-1} x_{\rho_{j+1}}^{\alpha_{j+1}}\cdots x_{\rho_k}^{\alpha_k}$$
$$= \sum_{\substack{j=1\\ \alpha_j>0}}^{k} \alpha_j P^{l-1}_{(\alpha_1,\dots,\alpha_{j-1})}(x_1,\dots,x_{l-1}) x_l^{\alpha_j-1} P^{N-l}_{(\alpha_{j+1},\dots,\alpha_k)}(x_{l+1},\dots,x_N). \tag{21}$$

These partial derivatives have the following useful property:

Fact 87. *Let $k,d,N\in\mathbb{N}$, $\alpha\in\mathcal{I}(k,d)$, $k\le N$. Then $\sum_{l=1}^N \frac{\partial P_\alpha^N}{\partial x_l}\in H^{d-1,k}(\mathbb{R}^N)$.*

PROOF.

$$\sum_{l=1}^{N}\frac{\partial P_\alpha^N}{\partial x_l}(x) = \sum_{l=1}^{N}\sum_{\substack{j=1\\ \alpha_j>0}}^{k} \alpha_j \sum_{\substack{1\le\rho_1<\dots<\rho_{j-1}<l\\ l<\rho_{j+1}<\dots<\rho_k\le N}} x_{\rho_1}^{\alpha_1}\cdots x_{\rho_{j-1}}^{\alpha_{j-1}} x_l^{\alpha_j-1} x_{\rho_{j+1}}^{\alpha_{j+1}}\cdots x_{\rho_k}^{\alpha_k}$$
$$= \sum_{\substack{j=1\\ \alpha_j>0}}^{k} \alpha_j \sum_{l=1}^{N} \sum_{\substack{\rho\in\mathcal{N}(k,N)\\ \rho_j=l}} x_{\rho_1}^{\alpha_1}\cdots x_{\rho_{j-1}}^{\alpha_{j-1}} x_{\rho_j}^{\alpha_j-1} x_{\rho_{j+1}}^{\alpha_{j+1}}\cdots x_{\rho_k}^{\alpha_k} = \sum_{\substack{j=1\\ \alpha_j>0}}^{k} \alpha_j P^N_{(\alpha_1,\dots,\alpha_{j-1},\alpha_j-1,\alpha_{j+1},\alpha_k)}(x).$$

□

We note that this fact does not hold with $\mathcal{I}^+(k,d)$ and the space of sub-symmetric polynomials in place of $\mathcal{I}(k,d)$ and $H^{d-1,k}(\mathbb{R}^N)$, and this is the sole reason for considering the spaces $H^{d,K}(\mathbb{R}^N)$.

For each $x\in\mathbb{R}^N$ we naturally identify $DP_\alpha^N(x)$ with the vector $\left(\frac{\partial P_\alpha^N}{\partial x_1}(x),\dots,\frac{\partial P_\alpha^N}{\partial x_N}(x)\right)\in\mathbb{R}^N$.

Fact 88. *Let $M,N,k,d\in\mathbb{N}$, $M>N$, $\alpha\in\mathcal{I}(k,d)$, $k\le N$, and $x\in\mathbb{R}^N$. Then $DP_\alpha^N(x)$ is a linear combination of vectors $DP_\beta^M(x)\restriction_N = \left(\frac{\partial P_\beta^M}{\partial x_1}(x),\dots,\frac{\partial P_\beta^M}{\partial x_N}(x)\right)\in\mathbb{R}^N$, $\beta\in\bigcup_{m=1}^k\mathcal{I}(m,d)$.*

PROOF. Let $1\le m\le k$ be such that $\alpha_m>0$ and $\alpha_{m+1}=\dots=\alpha_k=0$. Fix $1\le l\le N$. If $\alpha_j>0$, then $m\ge j$ and hence by Fact 85

$$P^{N-l}_{(\alpha_{j+1},\dots,\alpha_k)}(x_{l+1},\dots,x_N) = \sum_{s=m}^{k} c_s P^{M-l}_{(\alpha_{j+1},\dots,\alpha_s)}(x_{l+1},\dots,x_N,0,\dots,0),$$

where $c_s = (-1)^{k-s}\binom{M-N+k-s-1}{k-s}$. Therefore using (21) and the fact that $\alpha_{s+1} = \dots = \alpha_k = 0$ if $m \leq s \leq k$ we obtain

$$\begin{aligned}\frac{\partial P^N_\alpha}{\partial x_l}(x) &= \sum_{\substack{j=1\\ \alpha_j>0}}^k \alpha_j P^{l-1}_{(\alpha_1,\dots,\alpha_{j-1})}(x_1,\dots,x_{l-1})x_l^{\alpha_j-1}\sum_{s=m}^k c_s P^{M-l}_{(\alpha_{j+1},\dots,\alpha_s)}(x_{l+1},\dots,x_N,0,\dots,0)\\ &= \sum_{s=m}^k c_s \frac{\partial P^M_{(\alpha_1,\dots,\alpha_s)}}{\partial x_l}(x),\end{aligned}$$

from which the statement follows.

□

We will also make use of the following version of the Lagrange multipliers theorem.

Theorem 89. *Let $G \subset \mathbb{R}^n$ be an open set, $f \in C^1(G)$, $F \in C^1(G;\mathbb{R}^m)$, and assume that F has a constant rank. If the function f has a local extremum with respect to $M = \{x \in G;\ F(x) = 0\}$ at $a \in M$, then $Df(a)$ is a linear combination of $DF_1(a),\dots,DF_m(a)$, where $F_1,\dots,F_m$ are the components of the mapping F.*

PROOF. Let $k = \operatorname{rank} F(x)$ for $x \in G$. Since DF is continuous, we may without loss of generality assume that $DF_1(x),\dots,DF_k(x)$ are linearly independent for each $x \in G$. From the Rank theorem it follows that there are C^1-smooth functions g_j of k variables, $j = k+1,\dots,m$, and a neighbourhood U of a such that $F_j(x) = g_j\big(F_1(x),\dots,F_k(x)\big)$ for each $x \in U$, $j = k+1,\dots,m$ (see e.g. [**Zorich:Analysis1**]). Notice that $g_j(0,\dots,0) = g_j\big(F_1(a),\dots,F_k(a)\big) = F_j(a) = 0$, $j = k+1,\dots,m$. Therefore $M \cap U = \{x \in U;\ F_1(x) = 0,\dots,F_k(x) = 0\}$ and we may use the classical version of the Lagrange multipliers theorem

□

Now we are ready to prove the key lemma.

Lemma 90. *For every $n, K \in \mathbb{N}$ there are $N \in \mathbb{N}$ and $u, v \in \mathbb{R}^N$ such that $P(u) = P(v)$ for every $P \in H^{n,K}(\mathbb{R}^N)$ but $s_{n+1}(u) \neq s_{n+1}(v)$.*

PROOF. The proof is based on the observation that $\sum_{l=1}^N \frac{\partial s_{n+1}}{\partial x_l}(x) = (n+1)s_n(x)$, which together with Fact 87 leads to an inductive proof. For each fixed $K \in \mathbb{N}$ we prove the statement by induction on n. So fix $K \in \mathbb{N}$ and denote $\mathcal{M}(n) = \bigcup_{1\leq d\leq n}\bigcup_{1\leq k\leq K} \mathcal{I}(k,d)$. The space $H^{n,K}(\mathbb{R}^N)$ is generated by a constant function and polynomials P^N_α, $\alpha \in \mathcal{M}(n)$. For $n = 1$ the functions P^N_α, $\alpha \in \mathcal{M}(n)$ are linear and so there is $N \in \mathbb{N}$ large enough such that $\bigcap_{\alpha\in\mathcal{M}(n)} \ker P^N_\alpha$ contains a non-zero element u. Then it suffices to take $v = 2u$.

The inductive step from $n-1$ to n will be proved by contradiction. So assume that for each $N \geq K$ and each $u, v \in \mathbb{R}^N$ satisfying $P^N_\alpha(u) = P^N_\alpha(v)$ for all $\alpha \in \mathcal{M}(n)$ we have $s_{n+1}(u) = s_{n+1}(v)$. Now let $F^N : \mathbb{R}^N \to \mathbb{R}^{|\mathcal{M}(n)|}$ be the mapping whose components are the polynomials P^N_α, $\alpha \in \mathcal{M}(n)$ in some fixed order and let $\mathbb{A}_N(x)$ be its Jacobi matrix at $x \in \mathbb{R}^N$, i.e. $\mathbb{A}_N(x) = \left(\frac{\partial P^N_\alpha}{\partial x_l}(x)\right)_{\substack{\alpha\in\mathcal{M}(n)\\ l=1,\dots,M}}$. Note that the number of rows of the matrix of functions $\mathbb{A}_N$ does not depend on N. Thus there is $N \geq K$ and $y \in \mathbb{R}^N$ such that $\operatorname{rank} \mathbb{A}_N(y) = r = \max_{M\geq K, x\in\mathbb{R}^M} \operatorname{rank} \mathbb{A}_M(x)$.

By the inductive hypothesis there are $M > N$ and $g, h \in \mathbb{R}^{M-N}$ such that $P(g) = P(h)$ for all $P \in H^{n-1,K}(\mathbb{R}^{M-N})$ but $s_n(g) \neq s_n(h)$. If we denote by $\mathbb{A}_M(x)\restriction_N$ the matrix consisting of the first N columns of the matrix $\mathbb{A}_M(x)$, then $r = \operatorname{rank} \mathbb{A}_N(y) \leq \operatorname{rank} \mathbb{A}_M(y)\restriction_N \leq \operatorname{rank} \mathbb{A}_M(y) \leq r$, where the first inequality follows from Fact 88. Let $w^M_1,\dots,w^M_r$ be the rows of $\mathbb{A}_M$ such that $w^M_1(y)\restriction_N,\dots,w^M_r(y)\restriction_N$ are linearly independent. Using the continuity of the entries of $\mathbb{A}_M$ it is easy to see that there is a neighbourhood $U \subset \mathbb{R}^M$ of y such that for each $x \in U$ the vectors $w^M_1(x)\restriction_N,\dots,w^M_r(x)\restriction_N$ are linearly independent and so they form a basis of the space spanned by the rows of $\mathbb{A}_M(x)\restriction_N$. Clearly the same holds for $w^M_1(x),\dots,w^M_r(x)$ and $\mathbb{A}_M(x)$.

Fix an arbitrary $z \in U$ and put $S = \{x \in U;\ P_\alpha^M(x) = P_\alpha^M(z), \alpha \in \mathcal{M}(n)\}$. By our assumption s_{n+1} is constant on S and so Theorem 89 implies that $Ds_{n+1}(z)$ is a linear combination of the rows of $\mathbb{A}_M(z)$. It follows that for each $z \in U$ the vector $Ds_{n+1}(z)$ is a linear combination of $w_1^M(z), \dots, w_r^M(z)$.

Next, we put $u = y + c\sum_{j=1}^{M-N} g_j e_{N+j}$, $v = y + c\sum_{j=1}^{M-N} h_j e_{N+j}$ for some suitable $c \neq 0$ so that $u, v \in U$. Notice that since $H^{n-1,K}(\mathbb{R}^{M-N})$ is generated by homogeneous polynomials, we still have $P(cg) = P(ch)$ for all $P \in H^{n-1,K}(\mathbb{R}^{M-N})$ but $s_n(cg) \neq s_n(ch)$. For a fixed $\alpha \in \mathcal{M}(n)$ and $1 \le l \le N$ consider the polynomial $P(x) = \frac{\partial P_\alpha^M}{\partial x_l}(y_1, \dots, y_N, x_1, \dots, x_{M-N})$. Then by (21) and Fact 86 we have $P \in H^{n-1,K}(\mathbb{R}^{M-N})$ and so $P(cg) = P(ch)$. Therefore

$$w_j^M(u)\restriction_N = w_j^M(v)\restriction_N, \quad j = 1, \dots, r. \tag{22}$$

We have $Ds_{n+1}(u) = \sum_{j=1}^r \lambda_j w_j^M(u)$ and $Ds_{n+1}(v) = \sum_{j=1}^r \mu_j w_j^M(v)$ for some $\lambda_j, \mu_j \in \mathbb{R}$ and of course the same holds when we restrict to the first N coordinates of all of these vectors. But since $Ds_{n+1}(u)\restriction_N = (n+1)(y_1^n, \dots, y_N^n) = Ds_{n+1}(v)\restriction_N$, combining this with (22) and the fact that $w_1^M(u)\restriction_N, \dots, w_r^M(u)\restriction_N$ are linearly independent we obtain $\mu_j = \lambda_j$, $j = 1, \dots, r$. Finally, from Fact 87 and Fact 86 it follows that $x \mapsto \sum_{l=1}^M w_j^M\big(y + \sum_{j=1}^{M-N} x_j e_{N+j}\big)_l \in H^{n-1,K}(\mathbb{R}^{M-N})$, $j = 1, \dots, r$. Therefore

$$(n+1)s_n(u) = \sum_{l=1}^M \frac{\partial s_{n+1}}{\partial x_l}(u) = \sum_{j=1}^r \lambda_j \sum_{l=1}^M w_j^M(u)_l = \sum_{j=1}^r \lambda_j \sum_{l=1}^M w_j^M(v)_l = \sum_{l=1}^M \frac{\partial s_{n+1}}{\partial x_l}(v) = (n+1)s_n(v).$$

Since $s_n(u) = s_n(y) + s_n(cg)$ and $s_n(v) = s_n(y) + s_n(ch)$, we get $s_n(cg) = s_n(ch)$, which is a contradiction. $\square$

Let X be a real Banach space. We are going to work with algebras of polynomials on X, i.e. linear subspaces of $\mathcal{P}(X)$ that are closed with respect to pointwise multiplication. Given a set $B \subset \mathcal{P}(X)$ we say that B generates an algebra $\mathcal{A} \subset \mathcal{P}(X)$ if $\mathcal{A}$ is the smallest algebra containing B, i.e. it is the intersection of all algebras containing B. It is easy to see that B generates $\mathcal{A}$ if and only if for every $p \in \mathcal{A}$ there is a subset $\{b_1, \dots, b_k\} \subset B$ and a polynomial $P \in \mathcal{P}(\mathbb{R}^k)$ such that $p = P \circ (b_1, \dots, b_k)$.

Corollary 91. *For every $n \in \mathbb{N}$ there exist $N \in \mathbb{N}$ and $\varepsilon > 0$ such that for every $M \ge N$*

$$\sup_{x \in B_{\ell_1^M}} |p(x) - s_{n+1}(x)| \ge \varepsilon$$

for every p from the algebra generated by the sub-symmetric polynomials on $\mathbb{R}^M$ of degree at most n.

PROOF. Applying Lemma 90 to $K = n$ we obtain $N \in \mathbb{N}$ and $u, v \in B_{\ell_1^N}$ such that $P(u) = P(v)$ for every $P \in H^{n,n}(\mathbb{R}^N)$ but $s_{n+1}(u) \neq s_{n+1}(v)$. We put $\varepsilon = \frac{1}{2}|s_{n+1}(u) - s_{n+1}(v)|$. Let $M \ge N$. Since all sub-symmetric polynomials from $\mathcal{P}^n(\mathbb{R}^N)$ are contained in $H^{n,n}(\mathbb{R}^N)$, from the remark after Fact 85 it follows that in particular $P(u) = P(v)$ for every sub-symmetric $P \in \mathcal{P}^n(\mathbb{R}^M)$. We conclude that $p(u) = p(v)$ for every p from the algebra generated by the sub-symmetric polynomials from $\mathcal{P}^n(\mathbb{R}^M)$. The statement now easily follows. $\square$

12. Polynomial algebras on Banach spaces

For every Banach space X we denote by $\mathcal{A}_n(X)$ the algebra generated by $\mathcal{P}^n(X)$. Generally speaking, with a single exception when $\mathcal{P}(X) = \mathcal{P}_{\mathrm{wu}}(X)$, there are no results giving a characterisation of the uniform closure $\overline{\mathcal{P}(X)}^{\tau_b}$ in any infinite-dimensional Banach space. The refinement of the problem is finding the characterisation of $\overline{\mathcal{A}_n(X)}$, and this is wide open as well. The results in this section focus on the natural question when $\overline{\mathcal{A}_n(X)} = \overline{\mathcal{A}_{n+1}(X)}$ (of course, the inclusion $\subset$ always holds). More precisely, we are going to use the theory of sub-symmetric polynomials developed in the previous section together with the asymptotic approach to polynomial behaviour to obtain rather general results showing that the inclusion $\supset$ is almost never satisfied.

We begin by formulating a positive result.

Theorem 92 ([**AroPro:PolyAppr**]). *Let X, Y be Banach spaces and $U \subset X$ an open convex set. Then $\overline{\mathcal{P}_{\mathrm{f}}(X;Y)}^{\tau_b} = \mathcal{C}_{\mathrm{wu}}(U,Y)$.*

PROOF. Let $f \in \mathcal{C}_{\mathrm{wu}}(U,Y)$, let $V \subset U$ be a CCB set, and $\varepsilon > 0$. By the assumption there exist $\delta > 0$ and $\phi_1,\dots,\phi_n \in B_{X^*}$ such that $\|f(x)-f(y)\| < \frac{\varepsilon}{2}$ whenever $x, y \in V$, $|\phi_j(x-y)| < 2\delta$ for all $j \in \{1,\dots,n\}$. Since f is uniformly continuous on V, there is $M > 0$ such that $\|f\|_V \le M$. Let $\Phi\colon X \to \mathbb{R}^n$ be defined as $\Phi(x) = \bigl(\phi_1(x),\dots,\phi_n(x)\bigr)$. We set $K = \overline{\Phi(V)}$. Consider $\mathbb{R}^n$ with the maximum norm. Since $\Phi(V)$ is relatively compact, there are points $y_1,\dots,y_m \in \Phi(V)$ such that $\{U(y_k,\delta)\}_{k=1}^m$ is a covering of K. By Lemma **??** there is a partition of unity $\{\psi_k\}_{k=1}^m \subset \mathcal{P}_{\mathrm{f}}(\mathbb{R}^n)$ on K satisfying $\psi_k(y) < \frac{\varepsilon}{4mM}$ whenever $y \in K \setminus U(y_k,2\delta)$, $k = 1,\dots,m$. Choose $x_k \in V$ such that $y_k = \Phi(x_k)$. Finally, put

$$P(x) = \sum_{k=1}^m \psi_k\bigl(\Phi(x)\bigr) f(x_k).$$

Obviously $P \in \mathcal{P}_{\mathrm{f}}(X;Y)$. To show that P approximates f on V fix any $x \in V$. Let $I = \{1 \le k \le m;\ \Phi(x) \in U(y_k,2\delta)\}$ and $J = \{1,\dots,m\} \setminus I$. Then

$$\begin{aligned}\|f(x)-P(x)\| &= \left\|\left(\sum_{k=1}^m \psi_k\bigl(\Phi(x)\bigr)\right) f(x) - \sum_{k=1}^m \psi_k\bigl(\Phi(x)\bigr) f(x_k)\right\| \le \sum_{k=1}^m \psi_k\bigl(\Phi(x)\bigr)\|f(x)-f(x_k)\| \\ &\le \sum_{k\in I} \psi_k\bigl(\Phi(x)\bigr)\|f(x)-f(x_k)\| + \sum_{k\in J} \psi_k\bigl(\Phi(x)\bigr)\bigl(\|f(x)\|+\|f(x_k)\|\bigr) \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

□

Proposition 93. *Let X be a Banach space such that it does not contain ℓ_1 and $\mathcal{P}({}^nX) = \mathcal{P}_{\mathrm{wsc}}({}^nX)$. Then*

$$\overline{\mathcal{A}_1(X)} = \overline{\mathcal{A}_2(X)} = \dots = \overline{\mathcal{A}_n(X)}.$$

PROOF. Clearly, $\mathcal{P}_{\mathrm{f}}(X) \subset \mathcal{A}_1(X)$ (see Definition **??**). Thus $\mathcal{P}({}^nX) = \mathcal{P}_{\mathrm{wu}}({}^nX) \subset \mathcal{C}_{\mathrm{wu}}(X) = \overline{\mathcal{P}_{\mathrm{f}}(X)}^{\tau_b} \subset \overline{\mathcal{A}_1(X)}$, where the first equality follows from Corollary **??** combined with our assumption and the second equality is Theorem **??**. Since $\overline{\mathcal{A}_1(X)}$ is an algebra, $\mathcal{A}_n(X) \subset \overline{\mathcal{A}_1(X)}$, from which the result follows.

□

By Corollary **??** and the classical Theorem **??**, the above proposition can be applied to all $C(K)$ spaces, K scattered compact, and all $n \in \mathbb{N}$. All our results going in the opposite direction will rely on the following criterion.

Theorem 94. *Let X be a Banach space and $n \in \mathbb{N}$ such that for each $\varepsilon > 0$ there is $P \in \mathcal{P}({}^nX)$ with the following property: For every $N \in \mathbb{N}$ there exists a linearly independent set $\{e_j\}_{j=1}^N \subset S_X$ such that*

$$\sup_{\sum_{j=1}^N |a_j| \le 1} \left| P\left(\sum_{j=1}^N a_j e_j\right) - \sum_{j=1}^N a_j^n \right| \le \varepsilon.$$

Then $\mathcal{P}({}^nX) \not\subset \overline{\mathcal{A}_{n-1}(X)}$.

PROOF. Denote by $S^n(\ell_1^m)$ the algebra generated by all sub-symmetric polynomials on ℓ_1^m of degree at most n. By Corollary 91 there is $m \in \mathbb{N}$ and $\varepsilon > 0$ such that $\|Q - s_n\| \ge 3\varepsilon$ for all $Q \in \overline{S^{n-1}(\ell_1^m)}$. Let $P \in \mathcal{P}({}^nX)$ be the polynomial from the assumptions of the theorem. We claim that $P \notin \overline{\mathcal{A}_{n-1}(X)}$.

By contradiction, suppose that there exist $P_1,\dots,P_k \in \mathcal{P}^{n-1}(X)$ and $r \in \mathcal{P}(\mathbb{R}^k)$ such that for $R = r \circ (P_1,\dots,P_k)$ we have $\|P - R\| < \varepsilon$. Put $K = 1 + \max_j \|P_j\|$ and let $0 < \eta \le 1$ be such that $|r(u) - r(v)| < \varepsilon$ whenever $u, v \in KB_{\ell_\infty^k}$, $\|u - v\|_{\ell_\infty^k} < \eta$. Using Theorem 57 recursively kn times we find $N \in \mathbb{N}$ such that for any linearly independent $\{e_j\}_{j=1}^N \subset S_X$ there exist $A \subset \{1,\dots,N\}$, $|A| = m$, and sub-symmetric polynomials $Q_1, \dots, Q_k \in \mathcal{P}^{n-1}(Y)$ such that $\|P_j \restriction_Y - Q_j\| < \eta$, $j = 1, \dots, k$, where $Y = \operatorname{span}\{e_j;\ j \in A\}$ with ℓ_1-norm.

Let $\{e_j\}_{j=1}^N$ be the linearly independent set from the assumptions of the theorem and $A \subset \{1, \dots, N\}$, $Q_1, \dots, Q_k \in \mathcal{P}^{n-1}(Y)$ as above. Note that since $\{e_j\}$ is normalised, $\|R\restriction_Y - P\restriction_Y\|_Y \le \|R\restriction_Y - P\restriction_Y\|_X < \varepsilon$. Put $Q = r \circ (Q_1, \dots, Q_k)$. Then $Q \in S^{n-1}(\ell_1^m)$ and $\|Q - s_n\| \le \|Q - R\restriction_Y\| + \|R\restriction_Y - P\restriction_Y\| + \|P\restriction_Y - s_n\| < 3\varepsilon$, a contradiction.

□

Theorem 95. *Let X be a Banach space that contains ℓ_1. Then*

$$\overline{\mathcal{A}_1(X)} \subsetneq \overline{\mathcal{A}_2(X)} \subsetneq \overline{\mathcal{A}_3(X)} \subsetneq \cdots$$

PROOF. Combine Proposition **??** and Theorem 94.

□

The following easy fact will be used later in the proofs.

Fact 96. *Let X be a Banach space and $\{x_n\} \subset X$. If there is a w^*-null sequence $\{\phi_n\} \subset X^*$ (in particular if X has a Schauder basis $\{(e_n; f_n)\}$ satisfying $\inf_n\|e_n\| > 0$ and $\{\phi_n\}$ is a subsequence of $\{f_n\}$) such that $\varepsilon = \inf_n|\phi_n(x_n)| > 0$, then $\{x_n;\ n \in \mathbb{N}\}$ is not relatively compact.*

PROOF. We construct a subsequence $\{x_{n_k}\}$ by induction. Put $n_1 = 1$. If n_k is already defined for some $k \in \mathbb{N}$, we find $n_{k+1} > n_k$ such that $|\phi_j(x_{n_k})| < \frac{\varepsilon}{2}$ whenever $j \ge n_{k+1}$. By our assumption $K = \sup_n\|\phi_n\| < +\infty$. The set $\{x_{n_k};\ k \in \mathbb{N}\}$ is then an $\frac{\varepsilon}{2K}$-separated set. Indeed, let $k, l \in \mathbb{N}$, $k < l$. Then $\|x_{n_l} - x_{n_k}\| \ge \frac{1}{K}\left|\phi_{n_l}(x_{n_l} - x_{n_k})\right| \ge \frac{1}{K}\left(|\phi_{n_l}(x_{n_l})| - |\phi_{n_l}(x_{n_k})|\right) \ge \frac{\varepsilon}{2K}$.

□

Now we prove another type of a diagonalisation result.

Lemma 97. *Let X be a Banach space with a Schauder basis $\{(x_n; f_n)\}$, Y a Banach space with a bounded shrinking Schauder basis, and $m \in \mathbb{N}$ such that $\mathcal{P}^{m-1}(X; Y^*) = \mathcal{P}_{\mathrm{K}}^{m-1}(X; Y^*)$. Let $P \in \mathcal{P}(^mX; Y^*)$ and denote by P_k the kth component of P, i.e. the composition of P with the kth coordinate functional on Y^*. Then for each $\varepsilon > 0$ there is a subsequence $\{x_{n_k}\}$ such that for every $k \in \mathbb{N}$*

$$\left|P_{n_k}(x) - a_k^m P_{n_k}(x_{n_k})\right| \le \frac{\varepsilon}{2^k}$$

whenever $x = \sum_{j=1}^\infty a_j x_{n_j} \in B_X$ is such that $\min\{j;\ a_j \neq 0\} \ge |\operatorname{supp} x|$.

PROOF. Note that we may assume without loss of generality that $\{x_n\}$ is normalised. For any polynomial $Q \in \mathcal{P}(X; Y^*)$ we will use the convention that Q_k denotes the kth component of Q, i.e. the composition of Q with the kth coordinate functional on Y^*.

For $\alpha \in \mathcal{J}(l, m)$, $q \in \mathbb{N}_0$ satisfying $q + |\alpha| \le m$, and $k \in \mathbb{N}$ we define polynomials $R_k^{\alpha,q} \in \mathcal{P}(^{m-|\alpha|-q}X)$ by

$$R_k^{\alpha,q}(x) = \binom{m}{\alpha, q, m-|\alpha|-q}\check{P}_k(^{\alpha_1}x_1, \dots, {}^{\alpha_l}x_l, {}^q x_k, {}^{m-|\alpha|-q}x).$$

Note that using Proposition 8 we obtain

$$P_k\left(\sum_{j=1}^\infty a_j x_j\right) = \sum_{\substack{\alpha \in \mathcal{J}(k-1,m)\\ 0 \le q \le m-|\alpha|}} a_1^{\alpha_1} \cdots a_{k-1}^{\alpha_{k-1}} a_k^q R_k^{\alpha,q}\left(\sum_{j=k+1}^\infty a_j x_j\right).$$

The proof of the theorem is two-stage. In the first step we discard the coordinates before k, in the second step we get rid of the coordinates after k (i.e. the polynomials $R_k^{\alpha,q}$).

Clearly $\{R_k^{\alpha,q};\ k \in \mathbb{N}, \alpha \in \mathcal{J}(l, m), 0 \le q \le m - |\alpha|, l \in \mathbb{N}\}$ is bounded in $\mathcal{P}^m(X)$. We claim that for fixed $\alpha \in \mathcal{J}(l, m)$ and $q \in \mathbb{N}_0$ with $q + |\alpha| \le m$ and $|\alpha| > 0$ we have $\lim_{k\to\infty}\|R_k^{\alpha,q}\| = 0$. Indeed, if this is not the case, then there is a sequence $\{v_j\} \subset B_X$ and a subsequence $\{R_{k_j}^{\alpha,q}\}$ such that $\inf_j|R_{k_j}^{\alpha,q}(v_j)| > 0$.

By passing to further subsequences we may also assume that $b_p = \lim_{j\to\infty} R^{\alpha,p}_{k_j}(v_j)$ exists finite for all $0 \le p \le m-|\alpha|$. Obviously $b_q \neq 0$ and so there is $t \in \mathbb{R}$ such that $\sum_{p=0}^{m-|\alpha|} b_p t^p \neq 0$.

Put $Q(x) = \check{P}({}^{\alpha_1}x_1, \dots, {}^{\alpha_l}x_l, {}^{m-|\alpha|}x)$. Then by our assumption $Q \in \mathcal{P}_{\mathrm{K}}({}^{m-|\alpha|}X; Y^*)$. By the Polarisation formula $(\check{P})_k = \check{P_k}$ and so $Q_k(tx_k + x) = \binom{m}{\alpha, m-|\alpha|}^{-1} \sum_{p=0}^{m-|\alpha|} t^p R^{\alpha,p}_k(x)$ by Lemma **??**. Hence $\lim_{j\to\infty} Q_{k_j}(tx_{k_j} + v_j) = \binom{m}{\alpha,m-|\alpha|}^{-1} \sum_{p=0}^{m-|\alpha|} b_p t^p \neq 0$. This however contradicts the compactness of Q by Fact 96.

Now we construct a subsequence $\{x_{m_k}\}$ so that

$$\left| P_{m_k}\left(\sum_{j=1}^{\infty} a_j x_{m_j}\right) - \sum_{q=0}^{m} a_k^q R^q_{m_k}\left(\sum_{j=k+1}^{\infty} a_j x_{m_j}\right)\right| \le \frac{\varepsilon}{3\cdot 2^k} \tag{23}$$

for any $\sum_{j=1}^{\infty} a_j x_{m_j} \in B_X$, where $R^q_k = R^{(0,\dots,0),q}_k$ for short. Let $K \ge 1$ be the basis constant of $\{x_n\}$. We set $m_1 = 1$. If $m_1, \dots, m_k$ are already defined, then we find $m_{k+1} > m_k$ so that $m|\mathcal{J}(k,m)|(2K)^m \|R^{\alpha,q}_{m_{k+1}}\| < \frac{\varepsilon}{3\cdot 2^{k+1}}$ for all $\alpha \in \mathcal{J}(m_k, m)$ such that $\alpha_j = 0$ if $j \notin \{m_1, \dots, m_k\}$ and $|\alpha| > 0$, and all $0 \le q \le m - |\alpha|$. It is easily checked that the sequence $\{m_k\}$ satisfies (23).

In the second step we show that the polynomials $R^q_{m_k}$, $q < m$ are asymptotically zero on a suitable subspace. Using induction we construct a subsequence $\{n_k\}$ of $\{m_k\}$ along with auxiliary nested subsequences $\{m_k\} \supset \{m^1_k\} \supset \{m^2_k\} \supset \cdots$. Set $n_1 = m_1$. By Theorem 66 there are a subsequence $\{m^1_k\}$ of $\{m_k\}$, a Banach space E with a Schauder basis $\{e_n\}$ that is a spreading model of $\{x_{m^1_k}\}$, and sub-symmetric polynomials $S^q_{n_1} \in \mathcal{P}({}^{m-q}E)$, $q = 0, \dots, m-1$, satisfying the following for $r = 1$:

$$\left| S^q_{n_r}\left(\sum_{j=1}^{N} a_j e_j\right) - R^q_{n_r}\left(\sum_{j=1}^{N} a_j x_{m^r_{l_j}}\right)\right| \le \frac{\varepsilon}{3m(2K)^m 2^r} \tag{24}$$

for all $N \le l_1 < l_2 < \cdots < l_N$, all scalars $a_1, \dots, a_N$ with $\sum_{j=1}^{N} a_j x_{m^r_{l_j}} \in B_X$, and all $0 \le q < m$. Now assume that $\{m^{r-1}_k\}$ is already defined for some $r \in \mathbb{N}$, $r > 1$. Then we put $n_r = m^{r-1}_r$ and again by Theorem 66 there are a subsequence $\{m^r_k\}$ of $\{m^{r-1}_k\}$ and sub-symmetric polynomials $S^q_{n_r} \in \mathcal{P}({}^{m-q}E)$, $q = 0, \dots, m-1$, satisfying (24). (Recall that subsequences of a sequence with a spreading model have the same spreading model.) Note that for each $k \in \mathbb{N}$ the sequence $\{n_j\}_{j=k+1}^{\infty}$ is a subsequence of $\{m^k_j\}_{j=1}^{\infty}$. It follows that

$$\left| S^q_{n_k}\left(\sum_{j=1}^{N} a_j e_j\right) - R^q_{n_k}\left(\sum_{j=1}^{N} a_j x_{n_{l_j}}\right)\right| \le \frac{\varepsilon}{3m(2K)^m 2^k}$$

for all $\max\{k+1, N\} \le l_1 < l_2 < \cdots < l_N$, all scalars $a_1, \dots, a_N$ with $\sum_{j=1}^{N} a_j x_{n_{l_j}} \in B_X$, $0 \le q < m$, and $k \in \mathbb{N}$. Combining these estimates with (23) and the fact that $P_k(x_k) = \sum_{q=0}^{m} R^q_k(0) = R^m_k(0)$ we obtain

$$\left|P_{n_k}(x) - f_{n_k}(x)^m P_{n_k}(x_{n_k})\right| \le \frac{2}{3}\frac{\varepsilon}{2^k} + \sum_{q=0}^{m-1} (2K)^q \left| S^q_{n_k}\left(\sum_{\substack{j=1\\ l_j \ge k+1}}^{N} a_j e_j\right)\right| \le \frac{2}{3}\frac{\varepsilon}{2^k} + (4K)^m \sum_{q=0}^{m-1} \|S^q_{n_k}\|$$

whenever $x = \sum_{j=1}^{N} a_j x_{n_{l_j}} \in B_X$ with $N \le l_1 < \cdots < l_N$ (assuming, as we may, that (17) in Proposition 65 holds with $\varepsilon_k \le \frac{1}{2}$).

We claim that $\lim_{k\to\infty} \|S^q_{m_k}\| = 0$ for each $0 \le q < m$. This finishes the proof by passing to a suitable subsequence of $\{n_k\}$ in the estimate above. By contradiction, assume that this does not hold for some $0 \le q < m$. Since $\{R^q_k;\ k \in \mathbb{N}\}$ is a bounded set, it follows that $\{S^q_{m_k}\}$ is a bounded sequence in a finite-dimensional space of sub-symmetric $(m-q)$-homogeneous polynomials on E. Thus $\{S^q_{m_k}\}$ has a non-zero cluster point S. In particular, there is a finitely supported $w = \sum_{j=1}^{r} w_j e_j \in \frac{1}{2} B_E$ such that $S(w) \neq 0$, which means that $\inf_k |S^q_{p_k}(w)| = 2\delta > 0$ for some subsequence $\{p_k\}$ of $\{m_k\}$. Put $w_l = \sum_{j=1}^{r} w_j x_{n_{l+j}}$

and note that $w_l \in B_X$ for l large enough. Then $\liminf_{l\to\infty}|R^q_{p_k}(w_l)| \geq \delta$ for each $k \in \mathbb{N}$ (dropping finitely many members of $\{p_k\}$ if necessary).

Now let $W^l(x) = \binom{m}{q}\check{P}({}^q x, {}^{m-q} w_l)$. Then $W^l \in \mathcal{P}({}^q X; Y^*) = \big((\otimes^q_{\pi,s} X) \otimes_\pi Y\big)^*$ (Corollary 36). Since $\{w_l\}$ is bounded, so is $\{W^l\}$. Since $(\otimes^q_{\pi,s} X) \otimes_\pi Y$ is separable, there is a subsequence $\{W^{l_j}\}$ such that $W^{l_j} \overset{w^*}{\to} W \in \mathcal{P}({}^q X; Y^*)$, and in particular $w^*\text{-}\lim_{j\to\infty} W^{l_j}(x_k) = W(x_k)$ for each $k \in \mathbb{N}$ (Corollary 36). Hence $|W_{p_k}(x_{p_k})| = \lim_{j\to\infty}|W^{l_j}_{p_k}(x_{p_k})| = \lim_{j\to\infty}|R^q_{p_k}(w_{l_j})| \geq \delta$ for each $k \in \mathbb{N}$. But according to our assumption the polynomial W is compact, which contradicts Fact 96.

□

Theorem 98. *Let X be a Banach space and let $m \in \mathbb{N}$ be such that there is a non-compact $P \in \mathcal{P}({}^m X; \ell_1)$. Then $\mathcal{P}({}^n X) \not\subset \overline{\mathcal{A}_{n-1}(X)}$ for every $n \geq m$.*

Proof. If X has a subspace isomorphic to ℓ_1, then the result follows from Theorem 95, so for the rest of the proof we assume that X does not contain ℓ_1. We may assume that m is minimal such that there is a non-compact $P \in \mathcal{P}({}^m X; \ell_1)$. Denote by $\{e_k\}$ the canonical basis of ℓ_1 and by P_k the kth component of P, i.e. the composition of P with the kth coordinate functional on ℓ_1. We may assume that there is a basic sequence $\{x_k\} \subset X$ such that $P(x_k) = e_k$ (although in the proof it suffices to have $\limsup|P_k(x_k)| > 0$). Indeed, by Lemma 42 there is a weakly null sequence $\{x_k\} \subset S_X$ such that $\{P(x_k);\ k \in \mathbb{N}\}$ is not relatively compact. By passing to a subsequence we may assume that $\{x_k\}$ is a normalised basic sequence such that $\{P(x_k)\}$ is not relatively compact. Then we may use Proposition **??**.

Denote by $\{f_k\}$ the functionals biorthogonal to $\{x_k\}$. Let $\varepsilon > 0$ and let $\{x_{n_k}\}$ be the subsequence from Lemma 97. Given $n \geq m$ we put

$$Q(x) = \sum_{k=1}^{\infty} f_{n_k}(x)^{n-m} P_{n_k}(x).$$

Then $Q \in \mathcal{P}({}^n X)$ (Theorem **??**) and it satisfies the condition laid out in Theorem 94. Indeed, given $N \in \mathbb{N}$ we put $y_j = x_{n_{N+j}}$, $j = 1, \dots, N$. If $x = \sum_{j=1}^N b_j y_j$ is such that $\sum_{j=1}^N |b_j| \leq 1$, then $x \in B_X$. We put $a_{N+j} = b_j$, $j = 1, \dots, N$ and $a_k = 0$ for $k \notin [N+1, N+N]$. Then $x = \sum_{k=1}^{\infty} a_k x_{n_k}$ and so

$$\left|Q(x) - \sum_{j=1}^N b_j^n\right| = \left|\sum_{k=1}^{\infty} a_k^{n-m} P_{n_k}(x) - \sum_{k=1}^{\infty} a_k^n P_{n_k}(x_{n_k})\right| \leq \sum_{k=1}^{\infty}\left|P_{n_k}(x) - a_k^m P_{n_k}(x_{n_k})\right| \leq \varepsilon.$$

□

Corollary 99. *Let X be a Banach space admitting a non-compact operator $T \in \mathcal{L}(X; \ell_p)$, $1 \leq p < \infty$. Put $n = \lceil p \rceil$. Then*

$$\overline{\mathcal{A}_{n-1}(X)} \subsetneqq \overline{\mathcal{A}_n(X)} \subsetneqq \overline{\mathcal{A}_{n+1}(X)} \subsetneqq \cdots . \tag{25}$$

Proof. By Proposition **??** we may assume that $T(B_X)$ contains the canonical basis of ℓ_p. It then suffices to compose T with the polynomial $P \in \mathcal{P}({}^n \ell_p; \ell_1)$ given by $P((x_j)_{j=1}^{\infty}) = (x_j^n)_{j=1}^{\infty}$ to obtain a non-compact n-homogeneous polynomial from X into ℓ_1. It remains to apply Theorem 98.

□

Corollary 100. *Let $X = L_p([0,1])$, $1 \leq p \leq \infty$, or $X = \ell_\infty$, or $X = C(K)$, K non-scattered compact. Then*

$$\overline{\mathcal{A}_1(X)} \subsetneqq \overline{\mathcal{A}_2(X)} \subsetneqq \overline{\mathcal{A}_3(X)} \subsetneqq \cdots$$

Proof. The spaces $L_1([0,1])$, ℓ_∞, $L_\infty([0,1])$, and $C(K)$, K non-scattered, contain ℓ_1 ([**FHHMZ**], Theorem **??**) and so Theorem 95 applies. The spaces $L_p([0,1])$, $1 < p < \infty$ contain a complemented subspace isomorphic to ℓ_2 ([**FHHMZ**]) and we may use Corollary 99.

□

Corollary 101. *Let $1 \leq p < \infty$ and $n = \lceil p \rceil$. Then*

$$\overline{\mathcal{A}_1(\ell_p)} = \cdots = \overline{\mathcal{A}_{n-1}(\ell_p)} \subsetneqq \overline{\mathcal{A}_n(\ell_p)} \subsetneqq \overline{\mathcal{A}_{n+1}(\ell_p)} \subsetneqq \cdots .$$

PROOF. By Proposition 93 and Corollary 49 we obtain $\overline{\mathcal{A}_{n-1}(\ell_p)} = \overline{\mathcal{A}_1(\ell_p)}$. The rest follows readily from Corollary 99.

□

Corollary 102. *Let X be a Banach space such that X^* is of type $p > 1$. Then*

$$\overline{\mathcal{A}_{n-1}(X)} \subsetneqq \overline{\mathcal{A}_n(X)} \subsetneqq \overline{\mathcal{A}_{n+1}(X)} \subsetneqq \cdots ,$$

where $n = [q] + 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF. Let $s \in (q, n)$. By Corollary **??** there is a normalised basic sequence $\{x_k\}_{k=1}^{\infty}$ in X and $T \in \mathcal{L}(X; \ell_s)$ such that $T(x_k) = e_k$, where $\{e_k\}$ is the canonical basis of ℓ_s. Thus T is not compact and an appeal to Corollary 99 finishes the argument.

□

Corollary 103 ([**DimGon:BlockPoly**])**.** *Let X be a Banach space with an unconditional basis that has no subspace isomorphic to ℓ_1, and suppose that n is the smallest integer such that there exists a $P \in \mathcal{P}(^nX)$ which is not weakly sequentially continuous. Then*

$$\overline{\mathcal{A}_1(X)} = \cdots = \overline{\mathcal{A}_{n-1}(X)} \subsetneqq \overline{\mathcal{A}_n(X)} \subsetneqq \overline{\mathcal{A}_{n+1}(X)} \subsetneqq \cdots$$

PROOF. The initial sequence of equalities follows from Proposition 93. On the other hand, by Lemma 41 we may assume that there is $a > 0$ and a normalised weakly null block sequence $\{x_j\}_{j=1}^{\infty}$ of the given unconditional basis such that $|P(x_j)| \geq a$. By Proposition 23 the space $\mathcal{P}(^nX)$ has a subspace isomorphic to ℓ_∞. To finish, we combine Theorems 43 and 98.

□

We remark that by only formal changes in the argument the result holds for any Banach space with an unconditional FDD (this is the formulation in [**DimGon:BlockPoly**]).