Ground states for dipolar quantum gases in the unstable regime

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(joint work with L. Jeanjean)
The dipolar Bose-Einstein condensate

The DBE is well described by the wave function $\psi(x, t)$ whose evolution is governed by the three-dimensional Gross-Pitaevskii equation (GPE)

$$ih \frac{\partial \psi(x, t)}{\partial t} = -\frac{h^2}{2m} \nabla^2 \psi + W(x) \psi + U_0 |\psi|^2 \psi + (V_{\text{dip}} \star |\psi|^2) \psi, \quad x \in \mathbb{R}^3, \quad t > 0.$$ 

Here $h$ is the Planck constant, $m$ is the mass of a dipolar particle and $W(x)$ is an external trapping potential, $W(x) = \frac{m}{2} a^2 |x|^2$ (a is the trapping frequency). $U_0 = 4\pi h^2 a_s / m$ is the local interaction, $a_s$ the $s$–wave scattering length. The long-range dipolar interaction is given by

$$V_{\text{dip}}(x) = \frac{\mu_0 \mu_{\text{dip}}^2}{4\pi} \frac{1 - 3\cos^2(\theta)}{|x|^3}, \quad x \in \mathbb{R}^3$$

where $\mu_0$ is the vacuum magnetic permeability, $\mu_{\text{dip}}$ is the permanent magnetic dipole moment and $\theta$ is the angle between the dipole axis (for simplicity we fix it $(0, 0, 1)$) and the vector $x$. 
The dipolar Bose-Einstein condensate

The wave function is normalized according to

\[ ||\psi||^2 := \int_{\mathbb{R}^3} |\psi(x, t)|^2 dx = N \]

where \( N \) is the total number of dipolar particles in the dipolar BEC.
The dimensionless GPE

We rescale the GPE into the following

\[ i \frac{\partial \psi(x, t)}{\partial t} = -\frac{1}{2} \nabla^2 \psi + \frac{a^2}{2} |x|^2 \psi + \lambda_1 |\psi|^2 \psi + \lambda_2 (K \star |\psi|^2) \psi, \quad x \in \mathbb{R}^3, \quad t > 0. \]

The dimensionless long-range dipolar interaction potential \( K(x) \) is given by

\[ K(x) = \frac{1 - 3 \cos^2(\theta)}{|x|^3}, \quad x \in \mathbb{R}^3. \]

The corresponding normalization is now

\[ N(\psi(\cdot, t)) := ||\psi(\cdot, t)||^2 = \int_{\mathbb{R}^3} |\psi(x, t)|^2 dx = \int_{\mathbb{R}^3} |\psi(x, 0)|^2 dx = 1 \]

and the physical parameters \((\lambda_1, \lambda_2)\), which describes the strength of the two nonlinearities.
The dimensionless GPE

The local existence and uniqueness of solutions to GPE in $H^1(\mathbb{R}^3)$ has been proved by Carles-Markowich-Sparber (’08). We focus on the case when $\lambda_1$ and $\lambda_2$ fulfills the following conditions

Unstable regime

\[
\begin{align*}
\lambda_1 - \frac{4}{3}\pi\lambda_2 &< 0, & \text{if } \lambda_2 > 0; \\
\lambda_1 + \frac{8}{3}\pi\lambda_2 &< 0, & \text{if } \lambda_2 < 0.
\end{align*}
\]

Why unstable?

Let us start focusing on the case $a = 0$, no trapping potential, and $\lambda_1 - \frac{4}{3}\pi\lambda_2 < 0$ (for simplicity). The energy is given by

\[
E(\psi) := \frac{1}{2} ||\nabla \psi||^2_2 + \frac{\lambda_1}{2} ||\psi||^4_4 + \frac{\lambda_2}{2} \int_{\mathbb{R}^3} (K \ast |\psi|^2)|\psi|^2 dx.
\]

There exists $\psi \in \Sigma$ such that $E(\psi) < 0$, and following the classical Virial argument (Glassey) easy to prove that $T^* < \infty$. 
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\[
E(\psi) := \frac{1}{2} \|\nabla \psi\|_2^2 + \frac{\lambda_1}{2} \|\psi\|_4^4 + \frac{\lambda_2}{2} \int_{\mathbb{R}^3} (K * |\psi|^2) |\psi|^2 dx.
\]

There exists $\psi \in \Sigma$ such that $E(\psi) < 0$, and following the classical Virial argument (Glassey) easy to prove that $T^* < \infty$. 
Why unstable?

Moreover it is easy to show that

\[ \inf \left\{ \psi \text{ s.t. } \|\psi\|_2^2 = c \right\} E(\psi) = -\infty \quad \text{for all } c > 0 \]

Take \( \psi^t(x) = t^{\frac{5}{4}} \psi(tx_1, tx_2, t^{\frac{1}{2}}x_3) \), and look what happens to \( E(\psi^t(x)) \).

Denoting the Fourier transform of \( \psi(x) \) by \( \mathcal{F}(\psi) := \int_{\mathbb{R}^3} \psi e^{-ix \cdot \xi} dx \) we notice that the Fourier transform of \( K \) is

\[ \hat{K}(\xi) = \frac{4}{3} \pi \left( \frac{2\xi_3^2 - \xi_1^2 - \xi_2^2}{|\xi|^2} \right) \in \left[ -\frac{4}{3} \pi, \frac{8}{3} \pi \right] \].

Then, thanks to the Plancherel identity, one gets

\[
\lambda_1 \|\psi\|^4 + \lambda_2 \int_{\mathbb{R}^3} (K \ast \psi^2) |\psi|^2 = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \lambda_1 + \lambda_2 \hat{K}(\xi) \right) |\hat{\psi}|^2 d\xi.
\]
Why unstable?

We have $||\psi^t||_2^2 = ||\psi||_2^2$ for all $t > 0$ and the energy rescales as

$$E(\psi^t) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla_{x_1, x_2} \psi|^2 dx + \frac{t}{2} \int_{\mathbb{R}^3} |\nabla_{x_3} \psi|^2 dx +$$

$$+ \frac{t^2}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \frac{4}{3} \pi \lambda_2 \frac{2t \xi_3^2 - t^2 \xi_1^2 - t^2 \xi_2^2}{t^2 \xi_1^2 + t^2 \xi_2^2 + t^2 \xi_3^2}) |\hat{\psi}|^2 d\xi.$$ 

We have that

$$\lim_{t \to \infty} \lambda_1 + \frac{4}{3} \pi \lambda_2 \frac{2t \xi_3^2 - t^2 \xi_1^2 - t^2 \xi_2^2}{t^2 \xi_1^2 + t^2 \xi_2^2 + t^2 \xi_3^2} = \lambda_1 - \frac{4}{3} \pi \lambda_2 < 0$$

which implies that $\lim_{t \to \infty} E(\psi^t) = -\infty$ thanks to Lebesgue’s theorem.
Qualitative picture: dark region is the unstable regime, the cone is the stable regime (energy bounded from below, GWP)
Standing waves

To find stationary states we make the ansatz

$$\psi(x, t) = e^{-i\mu t} u(x), \quad x \in \mathbb{R}^3$$

with the corresponding stationary equation

$$-\frac{1}{2} \Delta u + \lambda_1 |u|^2 u + \lambda_2 (K * u^2) u + \mu u = 0. \quad (1)$$

Antonelli-Sparber approach (a lâ Weinstein)

Minimize the following scaling invariant functional

$$J(v) := \frac{\|\nabla v\|^3}{2\|v\|^2} - \lambda_1 \|v\|^4 - \lambda_2 \int_{\mathbb{R}^3} (K * |v|^2) |v|^2.$$  

and rescale the minimizer.
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Standing waves: remark

Prove existence of minimizer for $J$ is not so simple as for the Weinstein functional

$$\tilde{J}(v) := \frac{||\nabla v||^3_2 ||v||_2}{||v||^4_4}.$$  

Here the functional is not rotational invariant. Minimizer are only Steiner symmetric (radial in $(x_1, x_2)$, symmetric with respect to $x_3$ axis), not radially symmetric.

What about stability?

For cubic focusing NLS the standing waves found by Weinstein are strongly unstable (close to $u$ there are initial data that blow up in finite time). Why? Two reasons: uniqueness of positive solution, variational characterization (Berestycki-Cavenave following again the Virial argument).
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Prove existence of minimizer for $J$ is not so simple as for the Weinstein functional

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The mountain pass approach

We directly work with $E(u)$ restricted to $S(c) = \{ u \in H^1 \text{ s.t. } ||u||_2^2 = c \}$, $(c = 1$ is the important one). Even if the $\inf_{S(c)} E(u) = -\infty$ there exists critical points ($\mu$ will be the Lagrange multiplier). $E(u)$ is again

$$
E(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2 \hat{K}(\xi)) |\hat{u}|^2 d\xi.
$$

In order to simplify the notation we define

$$
A(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad B(u) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2 \hat{K}(\xi)) |\hat{u}|^2 d\xi.
$$

$$
Q(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2 \hat{K}(\xi)) |\hat{u}|^2 d\xi.
$$

notice that

$$
\frac{d}{d\lambda} E(\lambda^{\frac{3}{2}} u(\lambda x)) |_{\lambda=1} = Q(u) \quad \text{(for all standing waves $Q = 0$)}
$$
The mountain pass geometry

Given $c > 0$ we say that $E(u)$ has a mountain pass geometry on $S(c)$ if there exists a $k_c > 0$ such that

$$
\gamma(c) := \inf_{g \in \Gamma(c)} \max_{t \in [0,1]} E(g(t)) > \max \{ \max_{g \in \Gamma(c)} E(g(0)), \max_{g \in \Gamma(c)} E(g(1)) \}
$$

holds in the set

$$
\Gamma(c) = \{ g \in C([0,1], S(c)), \ g(0) \in A_{k_c}, E(g(1)) < 0 \},
$$

where

$$
A_{k_c} = \{ u \in S(c) : \| \nabla u \|_2^2 \leq k_c \}.
$$
The mountain pass geometry
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The mountain pass geometry induces the existence of a Palais-Smale sequence at the level $\gamma(c)$. Namely a sequence $(u_n) \subset S(c)$ such that

$$E(u_n) = \gamma(c) + o(1), \quad \|E'|_{S(c)}(u_n)\|_{H^{-1}} = o(1).$$

If one can show $H^1$ boundness and in addition the compactness of $(u_n)$, namely that up to a subsequence, $u_n \to u$ in $H$, then a critical point is found at the level $\gamma(c)$.

Theorem [B.-Jeanjean ('14)]

Let $c > 0$ and assume unstable regime. Then $E(u)$ has a mountain pass geometry on $S(c)$ and there exists a couple $(u_c, \mu_c) \in H^1 \times \mathbb{R}^+$ solution with $\|u_c\|_2^2 = c$ and $E(u_c) = \gamma(c)$. In addition $u_c$ is a ground state in the sense that

$$E(u_c) = \inf\{E(u), u \in S(c), E'|_{S(c)}(u) = 0\}.$$
The mountain pass geometry

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Idea of the proof

Main ingredients:

- $\gamma(c) = \inf_{u \in V(c)} E(u)$ where

$$V(c) = \{ u \in S(c) : Q(u) = 0 \} ,$$

- For all $c_1 \in (0, c)$, $\gamma(c_1) > \gamma(c)$.

Importance of $\gamma(c) = \inf_{u \in V(c)} E(u)$

Ghoussoub-Preiss (’89) states the following localization: if $F$ is a closed set such that

$$F \cap \{ E \geq \gamma \}$$

disconnects $A_{k_c}$ from $E(g(1))$ then a specific sequence $u_n$ can be defined such that

$$E(u_n) = \gamma(c) + o(1), \quad \|E_{S(c)}'(u_n)\|_{H^{-1}} = o(1), \quad dist(u_n, F) = o(1).$$
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Main ingredients:

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Importance of $\gamma(c) = \inf_{u \in V(c)} E(u)$

We can choose

$$F = V(c),$$

and easily show that the Palais-Smale sequence fulfills $Q(u_n) = o(1)$. Moreover noticing that

$$E(u) - \frac{1}{3} Q(u) = \frac{1}{6} A(u)$$

we obtain boundness.

Importance of monotonicity of $\gamma(c)$

Consider the MP sequence $u_n$, then $u_n \rightharpoonup \bar{u} \neq 0$ (easy). The goal is show that $\bar{u} \in S(c)$. Notice that

$$\frac{1}{2} A(u_n - \bar{u}) + \frac{1}{2} A(\bar{u}) + \frac{1}{2} B(u_n - \bar{u}) + \frac{1}{2} B(\bar{u}) = \gamma(c) + o(1).$$

Remember that $\bar{u} \in V(c_1)$ and that $\gamma(c_1) = \inf_{u \in V(c_1)}$. 
Importance of $\gamma(c) = \inf_{u \in V(c)} E(u)$

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Importance of monotonicity of $\gamma(c)$

Moreover

$$Q(u_n - \bar{u}) + Q(\bar{u}) = Q(\bar{u}_n) + o(1)$$

and therefore

$$E(u_n - \bar{u}) + \gamma(c_1) \leq \gamma(c) + o(1).$$

and

$$\frac{1}{6} A(u_n - \bar{u}) = E(u_n - \bar{u}) - \frac{1}{3} Q(u_n - \bar{u}).$$

This implies $\bar{u} \in S(c)$ and $E(\bar{u}) = \gamma(c)$.

Natural constraint

Let $c > 0$ be arbitrary, then $V(c)$ is a natural constraint, i.e each critical point of $E|_{V(c)}$ is a critical point of $E|_{S(c)}$. 
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Instability Theorem

Denote the set of minimizers of $E(u)$ on $V(c)$

$$\mathcal{M}_c := \{ u_c \in V(c) : E(u_c) = \inf_{u \in V(c)} E(u) \}. $$

For any $u \in \mathcal{M}_c$ the standing wave $e^{-i\mu_c t}u_c$ where $\mu_c > 0$ is the Lagrange multiplier, is strongly unstable.

Remark

Any constrained critical point belongs to $\Sigma$, the variational characterization of $MP$ energy level permits to apply the Glassey argument.

$$\frac{d^2}{dt^2} \| x(v)(t) \|_2^2 = 2Q(v)$$

and looking at the set

$$\Theta = \left\{ v \in \Sigma, \ E(v) < E(u_c), \ |v|^2 = |u_c|^2, \ Q(v) < 0 \right\}. $$
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Comparison with Antonelli-Sparber solution

The solution (fixing the $L^2$ norm) obtained a lá Weinstein is at MP energy level.

Call $u$ the minimizer of $J(v)$ that solves for $\mu > 0$

$$-\frac{1}{2}\Delta u + \lambda_1 |u|^2 u + \lambda_2 (K \ast u^2) u + \mu u = 0$$

and set $||u||_2^2 = c$. $Q(u) = 0$ such that

$$J(u) = \frac{1}{4} 6^{3/2} c^{1/2} E^{1/2}(u).$$

If $u$ is not a minimizer of $E(u)$ on $V(c)$, the for $u_0$ minimizer of $E(u)$ on $V(c)$ we have again

$$J(u_0) = \frac{1}{4} 6^{3/2} c^{1/2} E^{1/2}(u_0).$$
Small data scattering when $a = 0$

We recall the Duhamel formula associated to the dipolar GPE

$$\psi(t) = U(t)\psi_0 - i\lambda_1 \int_0^t U(t-s)(|\psi|^2\psi)(s)ds - i\lambda_2 \int_0^t U(t-s)((K*|\psi|^2)\psi)(s)ds$$

where

$$U(t) = e^{it\Delta^{\frac{\Lambda}{2}}}$$

generates the time evolution of the linear Schrödinger equation, and the Strichartz estimates in $\mathbb{R}^d$, $d \geq 3$

$$\|U(\cdot)\varphi\|_{L_t^qL_x^r} \leq C\|\varphi\|_{L^2}$$

$$\|\int_0^t U(t-s)F(s)ds\|_{L_t^qL_x^r} \leq C\|F\|_{L_t^{q_1}L_x^{r_1}}$$

where the pairs $(q, r)$, $(q_1, r_1)$ are admissible, i.e $2 \leq r \leq \frac{2d}{d-2}$ and

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right)$$ (analogous for $(q_1, r_1)$).
Small data scattering when \( a = 0 \)

Let \( \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\} \). There exists \( \delta > 0 \) such that if \( ||\psi_0||_{H^1(\mathbb{R}^3)} < \delta \) then \( \psi(t) \) scatters in \( H^1(\mathbb{R}^3) \). More precisely there exist \( \psi_\pm \) such that

\[
\lim_{t \to \pm \infty} ||\psi(t) - e^{it\frac{\Delta}{2}} \psi_\pm||_{H^1(\mathbb{R}^3)} = 0.
\]

Idea of the proof

The idea is, as usual, to find some \( L^p_t W^{1,q}_x \) Strichartz admissible norm that is uniformly bounded in time. Crucial point is that \( K \) is \( L^p - L^p \) continuous, i.e. \( ||K \ast f||_p \leq c ||f||_p \) if \( 1 < p < \infty \) (Calderón-Zigmund). First we show that \( ||\psi_0||_{H^1(\mathbb{R}^3)} < \delta \) implies \( \sup_t ||\psi_0||_{H^1(\mathbb{R}^3)} < \infty \). Second we choose \( (p, r) = \left(\frac{8}{3}, 4\right) \) and we want to prove that

\[
||\psi||_{L^\frac{8}{3}_t W^{1,4}_x} \leq c ||\psi_0||_{H^1} + c ||\psi||_{L^\frac{5}{3}_t W^{1,4}_x}.
\]
Small data scattering when \( a = 0 \)

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\]
Small data scattering when $a = 0$

By using the Duhamel formula we have

$$\|\psi\|_{L_t^3 W_x^{1,4}}^8 \leq \|U(t)\psi_0\|_{L_t^3 W_x^{1,4}}^8 + \lambda_1 \| \int_0^t U(t-s)(|\psi|^2\psi)(s)ds\|_{L_t^3 W_x^{1,4}}^8 +$$

$$+ \lambda_2 \| \int_0^t U(t-s)((K \star |\psi|^2)\psi)(s)ds\|_{L_t^3 W_x^{1,4}}^8.$$

Using Strichartz estimates (??), (??) we get

$$\|\psi\|_{L_t^3 W_x^{1,4}}^8 \leq c \|\psi_0\|_{H^1}^8 + c \||\psi|^2\psi\|_{L_t^5 W_x^{1,4/3}}^8 +$$

$$+ c \| (K \star |\psi|^2)\psi \|_{L_t^5 W_x^{1,4/3}}^8.$$
Small data scattering when $a = 0$

Let us look at the term $\| (K \ast |\psi|^2) \psi \|_{L_t^5 L_x^3}^{\frac{4}{5}} L_t^4 L_x^4$.

$$\| (K \ast |\psi|^2) \psi \|_{L_t^5 L_x^3}^{\frac{4}{5}} \leq c \| \psi \|_{L_t^3 L_x^4} \| K \ast |\psi|^2 \|_{L_t^4 L_x^2} \leq c \| \psi \|_{L_t^3 L_x^3} \| \psi \|_{L_t^5 L_x^3}^{\frac{4}{5}} L_t^4 L_x^4.$$  

Now using $\sup_t \| \psi_0 \|_{H^1(\mathbb{R}^3)} < \infty$ and Sobolev embedding

$$\| \psi \|_{L_t^8 L_x^4}^2 \leq \| \psi \|_{L_t^3 L_x^4}^{\frac{2}{3}} L_t^4 L_x^4 \leq c \| \psi \|_{L_t^3 L_x^4}^{\frac{4}{3}} L_t^8 L_x^4 \leq c \| \psi \|_{L_t^3 L_x^4}^{\frac{4}{3}} L_t^8 L_x^4 \leq c \| \psi \|_{L_t^3 L_x^4}^{\frac{4}{3}} L_t^8 L_x^4.$$  

we obtain

$$\| (K \ast |\psi|^2) \psi \|_{L_t^5 L_x^3}^{\frac{4}{5}} \leq c \| \psi \|_{L_t^3 L_x^4}^{\frac{5}{3}}.$$  

Similar computation holds for the other terms (where we use that $\| K \ast f \|_p \leq c \| f \|_p$).
Small data scattering when $a = 0$

Calling $||\psi||_{L^3_t W^{1,4}_x} = y$ and $||\psi_0||_{H^1} = b$ and looking at the function

$$f(y) = y - b - y^3$$

we notice that if $b$ is sufficiently small then

$$\{y \text{ s.t. } f(y) \leq 0\}$$

has two connected components. This implies that choosing $||\psi_0||_{H^1}$ sufficiently small we obtain

$$||\psi||_{L^3_t W^{1,4}_x} \leq K.$$ 

To conclude it is enough to show that $U(-t)\psi(t) \to \psi_+$ in $H^1$. Notice that for $0 < t < \tau$ and calling $g := \lambda_1 |\psi|^2 \psi + \lambda_2 (K * |\psi|^2) \psi$ and

$v(t) := U(-t)\psi(t)$, one gets

$$||v(t) - v(\tau)||_{H^1} \leq c ||g(\psi)||_{L^5_{[t,\tau]} W^{1,\frac{4}{3}}_x} \to_{t,\tau \to \infty} 0.$$
Small data scattering when $a = 0$

Calling $\|\psi\|_{L_t^\frac{8}{3}W_x^{1,4}} = y$ and $\|\psi_0\|_{H^1} = b$ and looking at the function

$$f(y) = y - b - y^\frac{5}{3}$$

we notice that if $b$ is sufficiently small then

$$\{ y \text{ s.t. } f(y) \leq 0 \}$$

has two connected components. This implies that choosing $\|\psi_0\|_{H^1}$ sufficiently small we obtain

$$\|\psi\|_{L_t^\frac{8}{3}W_x^{1,4}} \leq K.$$

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$$\|v(t) - v(\tau)\|_{H^1} \leq c\|g(\psi)\|_{L_t^\frac{8}{3}W_x^{1,4}} \to t,\tau \to \infty 0.$$
Standing waves in case \( a \neq 0 \), trapping potential active.

When \( a > 0 \) the functional becomes

\[
E_a(u) := \frac{1}{2} ||\nabla u||_2^2 + \frac{a^2}{2} ||x||_2^2 + \frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2 \hat{K}(\xi))|\hat{u}|^2 d\xi.
\]

Here we have a change of geometry of the constrained energy functional due to the Heisenberg uncertainty principle,

\[
\left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^3} |x|^2 |u|^2 dx \right)^\frac{1}{2} \geq \frac{3}{2} \left( \int_{\mathbb{R}^3} |u|^2 dx \right).
\]

The energy functional \( E_a(u) \), thanks to Gagliardo-Nirenberg inequality, fulfills

\[
E_a(u) \geq \frac{1}{2} A(u) + \frac{9a^2 c^2}{8A(u)} + \frac{1}{2} B(u) \geq \frac{1}{2} A(u) + \frac{9a^2 c^2}{8A(u)} - CA(u)^{\frac{3}{2}} c^{\frac{1}{2}}
\]
The geometry functional when $a \neq 0$. 

![Graph showing energy vs norm for different cases with $a \neq 0$.]
Theorem [B.-Jeanjean (’14)]

Let $c > 0$, then there exists $a_0 = a_0(\lambda_1, \lambda_2)$ such that for any $a \in ]0, a_0]$ the functional $E_a(u)$ restricted to $S(c)$ admits two critical points. One, denoted $u_a^1$, is a local minimizer and the second one $u_a^2$ corresponds to a mountain pass critical level. In addition

1. $u_a^1$ and $u_a^2$ are real, non negative, radially symmetric in the $(x_1, x_2)$ plane and axially symmetric with respect to the $x_3$ axis.
2. For any $a \in ]0, a_0]$, $0 < E_a(u_a^1) < E_a(u_a^2)$.
3. As $a \to 0$ we have $E_a(u_a^1) \to 0$ and $E_a(u_a^2) \to \gamma(c)$, where $\gamma(c)$ is the least energy level of $E$, the functional without the trapping potential.
Some comments

Boundness of local minimizing sequence follows for free. Compactness by the fact that the embedding $\Sigma \hookrightarrow L^p(\mathbb{R}^3)$ is compact for $p \in [2, 6)$.

Boundness of MP sequence cannot follow the Ghoussoub-Preiss argument. Here we don’t know if

$$V_a(c) \cap \{E_a \geq \gamma_a\}$$

where

$$V_a(c) = \left\{ u \in S(c), \quad Q_a(u) := A(u) - a^2 \int |x|^2 u^2 dx + \frac{3}{2} B(u) = 0 \right\}$$

disconnects (for sure $\gamma_a(c) > \inf_{V_a} E_a(u)$). We shall use a technical localization lemma (Louis idea) that permits to show (as in case $a = 0$) that $Q_a(u_n) = o(1)$. Boundness follows from

$$E_a(u) - \frac{1}{3} Q_a(u) = \frac{1}{6} A(u) + \frac{5}{6} a^2 D(u)$$
No small data scattering when $a \neq 0$

Consider the ground state energy for the quantum harmonic oscillator, minimize

$$F_{GS}(c) = \min_{S(c)} \frac{1}{2} \|\nabla u\|^2 + \frac{a^2}{2} \|x|u|^2$$  \hspace{1cm} (4)

We have from Heisenberg uncertainty principle written in the following form

$$\|\nabla u\|^2 + \omega^2 \|x|u|^2 - 3\omega \|u\|^2 \geq 0 \quad \forall u \in \Sigma, \omega > 0$$

that $F_{GS}(c) = \frac{3}{2} ac$ (no scattering for small data for quantum harmonic oscillator). The same argument works for the local minimizers found before (no scattering for small data when $a$ is small)
Final remarks

- The standing wave corresponding to the local minimum \( u_a^1 \) is orbitally stable (in the unstable regime!).
- There is a gap in the ground state energy of stationary states when \( a \to 0 \).
- Ground states when \( a = 0 \) have \( \mu > 0 \), ground states when \( a \neq 0 \) and small have \( \mu < 0 \).

Open questions

- Nondegeneracy when \( a = 0 \) of linearized equation?
- When \( a = 0 \) and \( u_0 \in H^1 \) be an initial condition with \( c = ||u_0||^2_2 \), if
  \[
  Q(u_0) > 0 \text{ and } E(u_0) < \gamma(c),
  \]
  then the solution of GPE with initial condition \( u_0 \) exists globally in times. Is there scattering as in Kenig-Merle?
Final remarks

- The standing wave corresponding to the local minimum $u_a^1$ is orbitally stable (in the unstable regime!).
- The is a gap in the ground state energy of stationary states when $a \to 0$
- Ground states when $a = 0$ have $\mu > 0$, ground states when $a \neq 0$ and small have $\mu < 0$

Open questions

- Nondegeneracy when $a = 0$ of linearized equation?
- When $a = 0$ and $u_0 \in H^1$ be an initial condition with $c = ||u_0||_2^2$, if
  \[ Q(u_0) > 0 \text{ and } E(u_0) < \gamma(c), \]
  then the solution of GPE with initial condition $u_0$ exists globally in times. Is there scattering as in Kenig-Merle?
THANK YOU FOR YOUR ATTENTION!