The progeny of a branching process

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Program

- Branching processes and their progeny
- The homographic examples
- Recovering the fecondity law from the progeny
- A difficult problem: is the Sibuya distribution a progeny?.

I: Branching process: one dimensional case

If p is a distribution on the set $\mathbb{N}$ of non negative integers with generating function

$$f_p(z) = \sum_{n=0}^{\infty} p_n z^n$$

then a branching process $(Z_n)_{n=0}^{\infty}$ governed by p is the Markov chain defined on $\mathbb{N}$ by $Z_0 = 1$ and

$$Z_{n+1} = \sum_{k=0}^{Z_n} X_{n,k}$$

where $(X_{n,k})_{n,k \geq 0}$ are iid random variables with distribution p. It is easily seen that

$$\mathbb{E}(z^{Z_n}) = f_p^{(n)}(z) = f_p \circ \cdots \circ f_p(z) \quad n \text{ times.} \tag{1}$$

I: Progeny: one dimensional case

If $(Z_n)_{n=0}^{\infty}$ is a branching process governed by p, a classical fact is that $\Pr(\exists n \ : \ Z_n = 0) = 1$ is and only if $m = \sum_{n=0}^{\infty} np_n \leq 1$. Under these circumstances the random variable

$$S = \sum_{n=0}^{\infty} Z_n$$

is finite. Its distribution q is called the progeny of p and we have the following link between the generating functions of p and q: for all z such that $|z| \leq 1$ the following holds

$$\boxed{f_q(z) = zf_p(f_q(z))} \qquad (2)$$

Since $Z_0 = 1$ the sum S is concentrated on $\mathbb{N}^+$.

I: Branching process: multitype

Given a sequence $p = (p_1, \dots, p_k)$ of probabilities on $\mathbb{N}^k$, the multivariate branching process $(Z_n)_{n\geq 0}$ governed by p is a Markov chain valued in $\mathbb{N}^k$ with transition probability defined as follows. With the notation $s^z = s_1^{z_1} \dots s_k^{z_k}$ we have

$$\mathbb{E}(s^{Z_{n+1}}|Z_n = z) = \prod_{i=1}^{k}[\sum_{j\in\mathbb{N}^k} p_i(j)s^j]^{z_i} = \prod_{i=1}^{k}[f_{p_i}(s)]^{z_i}.$$

In other terms if we have z_i particles of type i at time n, each particle of type i produces j_1 particles of type 1, j_2 particles of type 2 and so on, according to the probability distribution $j \mapsto p_i(j)$.

I: Progeny for multitype

If M is the (k,k) matrix of the means of Z_1 for $Z_0 = e_1, \dots, e_k$ its largest positive eigenvalue ρ exists and the necessary and sufficient condition for $\Pr(\lim Z_n = 0) = 1$ for any $Z_0 = z_0$ is $\rho \leq 1$. Therefore, suppose now that p is such that $\Pr(\lim Z_n = 0) = 1$ and define $S_{z_0} = \sum_{n=0}^{\infty} Z_n$. when $Z_0 = z_0$. Finally if $e_1, \dots, e_k$ is the canonical basis of $\mathbb{R}^k$ denote by q_i the distribution of S_{e_i}. The sequence $q = (q_1, \dots, q_k)$ is called the progeny of p. A simple probability reasoning shows that

$$f_q(s) = \mathrm{diag}(s_1, \dots, s_k) f_p(f_q(s)).$$

I: Examples of calculation of progeny in the univariate case

Given f_p we have to find $u = f_q(s)$ such that $u - sf_p(u) = 0$. The Lagrange formula says that

$$f_q(s) = \sum_{n=1}^{\infty} \frac{s^n}{n!} \left[\left(\frac{d}{du} \right)^{n-1} f_p(u)^n \right]_{u=0}$$

In rare cases it can be explicit. For Poisson $f_p(s) = e^{a(s-1)}$ with $a \leq 1$ to insure a mean ≤ 1 we get

$$f_q(s) = \sum_{n=1}^{\infty} \frac{s^n}{n!} (an)^{n-1} e^{a-n}.$$

For negative binomial $f_p(s) = q^a/(1-ps)^{-a}$, with $p(1+a) \leq 1$, we have

$$f_q(s) = \sum_{n=1}^{\infty} \frac{s^n}{n!} q^{an} p^{n-1} (an)_{n-1}.$$

(The Pochhammer symbol $(c)_k = c(c+1)(c+2)\dots(c+k-1)$ is used here)

Progenies and natural exponential families (NEF)

Proposition 1 Let f_p be governing a branching process with mean $m \leq 1$ and with generating function f_q for its progeny. Consider $r > 0$ such that $\boxed{f_{p,r}(s) = f_p(rs)/f_p(r)}$ exists and such that $m_r = rf_p'(r)/f_p(r) \leq 1$. Then the progeny $f_{q,a}(s)$ associated to $f_{p,r}$ is

$$\boxed{f_{q,a}(s) = f_q(as)/f_q(a) \quad \text{with} \quad r = f_q(a).}$$

n other terms, if the branching process is governed by a distribution belonging to the NEF generated by the probability p, the corresponding progeny has a distribution belonging to the NEF generated by q, but not with the same parameter.

Proof. $sf_{p,r}(f_{q,a}(s)) = \frac{s}{f_p(r)} f_p(rf_{q,a}(s)) = \frac{s}{f_p(r)} f_p(r\frac{f_q(as)}{f_q(a)})$

$$= \frac{as}{af_p(f_q(a))} f_p(f_q(as)) = \frac{f_q(as)}{f_q(a)}. \ \square$$

I: Example: the geometric distribution

For $\alpha \leq \frac{1}{2}$ consider $f_p(s) = \frac{1-\alpha}{1-\alpha s}$ Then $f_{p,r}(s) = \frac{1-\alpha r}{1-\alpha r s}$ when $r \leq \frac{1}{2}\alpha$ and

$$f_q(s) = \frac{1}{2\alpha}(1-\sqrt{1-4\alpha(1-\alpha)s}),\ f_{q,a}(s) = \frac{1-\sqrt{1-4\alpha(1-\alpha)as}}{1-\sqrt{1-4\alpha(1-\alpha)a}}.$$

with $r = f_q(a)$ or $a = \frac{r(1-\alpha r)}{1-\alpha}$.

(a formula similar to Proposition 1 in the multivariate case could be written)

II: Homographies in the univariate case

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $f_p(s) = h_M(s) = (as+b)/(cs+d)$, then the n^{th} iterate of f_p est h_{M^n}. Note that since $h_M = h_{\lambda M}$ and since $h_M(1) = 1$ we have 2 parameters instead of 4. We parameterize M as

$$M(p,r) = \begin{bmatrix} p+r-1 & 1-r \\ p-1 & 1 \end{bmatrix}$$

(the eigenvalues are p and r in $[0,1]$) then $M^n(p,r)$ is proportional to $M(p_n, r_n)$, where $p_n = p^n/\lambda_n$, $r_n = r^n/\lambda_n$, and $\lambda_n = (p^n(1-r) - r^n(1-p))/(p-r)$.
If $p = r$ (or $m = 1$), then $M^n(p,p)$ is proportional to $M(p_n, p_n)$, where $p_n = p/(p + n(1-p))$. In general we have the expansion

$$h_{M(p,r)}(s) = (1-r) + \sum_{n=0}^{\infty} rp(1-p)^{n-1}s^n.$$

II: Progenies for homographies in the univariate case

If $p \geq r$ (to insure extinction) then the progeny f_q is given by

$$f_q(s) = \frac{1}{2(1-p)}(1 - vs - \sqrt{\Delta(s)}),$$

where $v = p + r - 1$ and $\Delta(s) = (1 + vs)^2 - 4prs$. The proof of these facts is amusing: since $f_q(s) = h_{M(p,r)}(f_q(s))$, we have just to solve a second degree equation for getting f_q.
In particular if $v = 0$ this is a Sibuya distribution

$$f_q(s) = \frac{1}{2(1-p)}(1 - \sqrt{1 - 4p(1-p)s}).$$

II: Explicit form of the progeny in the general homographic case

If $v \neq 0$ the difficulty is in the expansion of

$$f_q(s) = \frac{1}{2(1-p)}(1 - vs - \sqrt{(1-sa)(1-sb)})$$

with $ab = v^2, \ a + b = 4pr - 2v$. I heard recently a lecture by V. Vinogradov where he states a formula equivalent to

$$\sum_{k=1}^{\infty} r^k \ {}_2F_1(\frac{k}{2}, \frac{k+1}{2}; 2, x) = \frac{2r}{x}(1 - r - \sqrt{(1-r)^2 - x})$$

and this formula leads to the coefficients of f_q. I have not seen his proof, but it can be also gotten from Letac and Piccioni (2015) Lemma 2, which says that if $U \sim \beta(\frac{1}{2}, \frac{1}{2}(d-1))$ is independent of $\epsilon \sim \frac{1}{2}(\delta_{-1} + \delta_1)$ then

$${}_2F_1(\frac{1}{2}p, \frac{1}{2}(p+1), d; x) = \mathbb{E}\left(\frac{1}{(1+\epsilon\sqrt{xU})^p}\right).$$

Suppose that we have a branching process of k types with the birth law exemplified for $k = 2$ by

$$\begin{aligned}\mathbb{E}(s^{Z_1}|Z_0 = (1,0)) &= \frac{a_{11}s_1 + a_{12}s_2 + b_1}{c_1 s_1 + c_2 s_2 + d},\\ \mathbb{E}(s^{Z_1}|Z_0 = (0,1)) &= \frac{a_{21}s_1 + a_{22}s_2 + b_2}{c_1 s_1 + c_2 s_2 + d}.\end{aligned}$$

Each function appearing in the components is called a fractional linear function, or homography. Note that the denominators are the same.

Let us consider a $(k+1, k+1)$ matrix G written in four blocks

$$G = \begin{bmatrix} A & b \\ c & d \end{bmatrix}$$

where A is a (k,k) matrix, where c is in E the space on line k vectors, b is in E^* the space of column k vectors and d is real. We shall denote by $\mathbf{1}$ the element of E^* whose entries are 1. Assume that the real number d is not 0 and consider the fractional linear

It is easily checked that the composition of two fractional linear mappings h_G and h_{G_1} satisfies $h_G \circ h_{G_1}(s) = h_{GG_1}(s)$ in a suitable small neighborhood of 0. This implies in particular that iterating the function h_G n times gives $h_G \circ h_G \circ \ldots \circ h_G = h_{G^n}$. Note that $h_G = h_{G_1}$ if and only if there exists a real number $\lambda \neq 0$ such that $G_1 = \lambda G$. If G is chosen such that each component of h_G is the generating function of a fractional linear distribution on $\mathbb{N}^k$, we shall say that h_G is the *generating function of a fractional linear birth law* and we consider the multitype branching process $Z = (Z_n)_{n \geq 0}$ such that

$$(\mathbb{E}(s^{Z_1}|Z_0 = e_i))_{i=1}^k = h_G(s).$$

Birth laws of this type are mentioned in Harris, page 49. The matrix of the means of this birth law is

$$M = \frac{1}{\langle c, \mathbf{1} \rangle + d}(A - \mathbf{1} \otimes c).$$

From Perron Frobenius, its highest positive eigenvalue ρ exists, and

II: Progenies for homographies in the multitype case

For $\rho \leq 1$ the progeny exists and

$$f_q(s) = \mathrm{diag}(s) h_G(f_q(s)).$$

This is equivalent to saying that there exists $\lambda \neq 0$ such that

$$\begin{bmatrix} \mathrm{diag}(s)A & \mathrm{diag}(s)b \\ c^T & d \end{bmatrix} \begin{bmatrix} f_q(s) \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} f_q(s) \\ 1 \end{bmatrix}. \qquad (3)$$

Here the eigenvalue λ must be $\langle c, f_q(s)\rangle + d$. We have seen that the solution is easy for $k = 1$. For $k = 2$ the explicit solution is too complex to be displayed here since the eigenvalue λ has the form

$$p_1 + (p_3 + p_6^{1/2})^{1/3} + (p_3 - p_6^{1/2})^{1/3},$$

where p_j is a non-homogeneous polynomial in (s_1, s_2) of degree j.

III: Recovering the fecundity law from the progeny

From the formula $f_q(s) = sf_p(f_q(s))$ suppose that we know f_q and suppose that we want f_p. If $u \in [0,1]$, there is a unique $s = g(u) = f^{(-1)}(u)$ which satisfies $u = f_q(s)$. Knowing g we recover f_p by

$$\boxed{f_p(u) = \frac{u}{g(u)}}.$$

Clearly, to be a progeny q must have a support S which is an additive semigroup of $\mathbb{N} \setminus \{0\}$. Also $1 \in S$- if not, $g'(0) = 0$ and $u/g(u)$ cannot be analytic. Thus $S = \mathbb{N} \setminus \{0\}$.
Similarly in the multivariate case we will get

$$\boxed{f_p(u) = (\mathrm{diag}(g(u))^{-1} u}.$$

III: Recovering the fecundity law from the progeny, continued

Reasonable necessary and sufficient conditions for q do not seem to be known. For instance I do not know when the translation by one of a Poisson distribution $f_q(s) = se^{a(s-1)}$ is a progeny. Here is a simpler case, the translation by one of a negative binomial law:

$$f_q(s) = \frac{s}{(2-s)^2}.$$

Then f_p should be

$$f_p(u) = \frac{u}{g(u)} = \frac{1}{8}(1 + 4u + \sqrt{1+8u}).$$

But this is not the generating function of a probability, since the coefficient of u^2 is -1.

III: Is the product of two progenies a progeny again? not always

This is a poorly raised question. More specifically if q_1 and q_2 are two progenies, is the function $s \mapsto \frac{f_{q_1}(s) f_{q_2}(s)}{s}$ the generating function of a new progeny q? For instance if $f_{p_1}(s) = \frac{1}{2-s}$ then its progeny is a Sibuya distribution since $f_{q_1}(s) = 1 - \sqrt{1-s}$ as seen before. But with $q_1 = q_2$ we get that $f_q(s) = \frac{(1-\sqrt{1-s})^2}{s}$ is the progeny of the binomial distribution $p = B(\frac{1}{2}; 2)$.
If we consider the case

$$f_q(s) = \frac{(1 - \sqrt{1-s})^3}{s^2}$$

we get after a careful calculation that f_p should be

$$f_p(u) = \frac{1}{16}((1 + 2u)\sqrt{1 + 8u} + 1 + 6u),$$

which is not a generating function since the coefficient of u^2 is -1.

IV: The Sibuya distribution

For given $a \in (0,1)$ the Sibuya distribution q_a is defined by its generating function

$$f_{q_a}(s) = 1-(1-s)^a = \sum_{n=1}^{\infty}\left(a\prod_{k=1}^{n-1}(k-a)\right)\frac{s^n}{n!}$$

It has some avatars, like the NEF extension for $r \in (0,1)$: $\frac{1-(1-rs)^a}{1-(1-r)^a}$ or the artificial addition of an atom $1-\lambda$ on zero : $1-\lambda(1-s)^a = (1-\lambda)+\lambda(1-(1-s)^a)$. But the reason of its popularity in branching process is the following fact:

$$f_{q_a}\circ f_{q_{a_1}} = f_{q_{aa_1}}.$$

In particular if $Z_1 \sim q_a$ then $Z_n \sim q_{a^n}$.

IV: Is the Sibuya distribution a progeny ?

We have met $f_{q_{\frac{1}{2}}}(s) = 1 - \sqrt{1-s}$ many times before not as a fecundity law, but as a progeny. The aim of this last section is to study the case of a general $a \in (0,1)$. It is a tough problem.

Theorem: q_a is a progeny if $\frac{1}{2} \leq a < 1$ and q_a is not a progeny if $\boxed{0 < a < \frac{1}{2} - 10^{-9}}$.

Conjecture: q_a is not a progeny if $\boxed{0 < a < \frac{1}{2}}$.

Proof of the theorem: For simplicity we denote $b = 1/a > 1$. $f_{q_a}(s) = u$ if and only if $s = g(u) = 1 - (1-u)^b$. So: when the power expansion of

$$\frac{u}{1-(1-u)^b}$$

has only non negative coefficients?

IV:The Sibuya distribution is a progeny if $1 < b \leq 2$.

$$\frac{u}{1-(1-u)^b} = \frac{1}{b} \times \frac{1}{1-uH(u)} = \frac{1}{b}\sum_{n=0}^{\infty} u^n H(u)^n. \qquad (4)$$

where

$$H(u) = \frac{b-1}{2} + \sum_{n=1}^{\infty}(b-1)(2-b)(3-b)\ldots(n+1-b)\frac{u^n}{(n+2)!}$$

Since $1 < b < 2$ all the coefficients of H are positive. as well as the coefficients of $\frac{u}{1-(1-u)^b}$.

IV:The case $b = 3$

If $b = 3$ here is a very simple proof that $1 - (1 - s)^{1/3}$ cannot be the generating function of a progeny. This comes from the fact that

$$\frac{1}{3} \times \frac{1}{1 - u + \frac{u^2}{3}} = \frac{1}{3} \sum_{n=0}^{\infty} r^n \frac{\sin(n+1)\theta}{\sin\theta} u^n,$$

where $re^{\pm i\theta} = \frac{1}{2}(3 \pm i\sqrt{3})$ are the complex roots of the polynomial $1 - u + \frac{u^2}{3}$. Actually $r = \sqrt{3}$ and $\theta = \pm\pi/6$. Clearly $\sin(n+1)\theta/\sin\theta \geq 0$ for all n is impossible and the result is proved. Note that $\sin(n+1)\theta/\sin\theta = 0$ for $n \equiv 5 \mod 6$.

IV: The case $b > 2$.

We begin with the following observation: if $\varphi_u(b) = \frac{bu}{1-(1-u)^b} - \frac{b-1}{2}u$ then $b \mapsto \varphi_u(b)$ is an even function, since it can be easily verified that $\varphi_u(b) + \varphi_u(-b) = 0$. As a consequence

$$\frac{bu}{1-(1-u)^b} = \frac{b-1}{2}u + \sum_{n=0}^{\infty} P_n(b)\frac{u^n}{n!}$$

where P_n is an even polynomial. Furthermore if $b = 1$ we have $h_1(u) = 1$. This implies that $P_n(1) = 1$ if $n \geq 1$. Therefore P_n is divisible by $b^2 - 1$.

IV: A study of the case $b > 2$

More specifically writing $X = (b^2 - 1)/3$ and $R_n(X) = 3 \times 2^{-n} X P_n(b)$ we obtain the following recurrence relation for the R_n's, correct for $n \geq 4$:

$$\boxed{(n+1)R_n(X) = 2(n-1)R_{n-1}(X) - X \sum_{k=2}^{n-2} R_k(X)R_{n-k}(X)},$$

with the initial conditions $R_2 = R_3 = 1$. Mathematica gives

$$R_4 = \frac{1}{5}(6 - X),\ R_5 = \frac{1}{5}(8 - 3X),\ R_6 = \frac{1}{35}(80 - 47X + 2X^2)$$

$$R_7 = \frac{1}{7}(24 - 19x + 2X^2),\ R_8 = \frac{1}{525}(2800 - 2754X + 488X^2 - 9X^3)$$

$$R_9 = \frac{1}{175}(-9X^3 + 188X^2 - 744X + 640) = \frac{1}{175}(X - 16)(9X^2 - 44X + 40).$$

IV: A study of the case $b > 2$, continued

The aim is now to show that
For any $X > 1$, there exists an n such that $R_n(X) < 0$.Note that $R_n(1) = 1$.

Proposition 2.3. For $n \geq 4$, the polynomial R_n has at least one zero in $(1, \infty)$. In particular, if x_n is the smallest zero > 1 of $R_n(X)$ for $n \geq 4$ then $1 < x_n < x_{n-1} < \ldots < x_4$. Furthermore $R_n(x_m) < 0$ for $m = n - 1$ and $m = n - 2$.

IV: A study of the case $b > 2$, continued

Proof. We prove the existence of x_n and the inequality $x_n < x_{n-1}$ by induction. We have

$$x_4 = 6 > x_5 = 8/3 > x_6 = 1,84... > x_7 = 3/2 > x_8 = 1.3162 > x_9 = 1.20$$

Suppose that $1 < x_{n-1} < \ldots < x_5 < x_4$ then for $k = 2, \ldots, n-2$ we can claim that $R_k(x_{n-1}) > 0$. As a consequence from (**??**) applied to $X = x_{n-1}$ we get that $R_n(x_{n-1}) < 0$. From the intermediate value theorem we get that x_n exists and that $x_n < x_{n-1}$. The induction is extended. Finally

$$(n+1)R_n(x_{n-2})$$

$$= 2(n-1)R_{n-1}(x_{n-2}) - x_{n-2}\sum_{k=3}^{n-3} R_k(x_{n-2})R_{n-k}(x_{n-2}) < 0. \square$$

IV: A study of the case $b > 2$, end

The conjecture will be proved if we show that the decreasing sequence $(x_n)_{n\geq 4}$ has limit 1. Mathematica shows that

$$\boxed{1 < x_{45} < 1 + \frac{1}{10000000000}.}$$

Thanks for your attention.