Embedding metric spaces into c_0.

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Let (M,d) and (X,d') be separable metric spaces.
$f : M \to X$ is a λ-embedding, if, for all $x, y \in M$,

$$d(x,y) \leq d'(f(x), f(y)) \leq \lambda d(x,y)$$

If moreover f is onto, then M and X are λ-Lipschitz equivalent.

Case $\lambda = 1$: 1-embedding = isometry.

Example : Suppose $f : \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz.
Then $F : \mathbb{R} \to (\mathbb{R}^2, \|\cdot\|_\infty)$ defined by $F(x) = \big(x, f(x)\big)$ is an isometry (non linear if $f(x) = \sin(x)$).

Let X and Y be Banach spaces and $f : X \to Y$ be an isometry such that $f(0) = 0$.

Figiel, 68 : If $\overline{vect}(f(X)) = Y$, then there exists $Q \in \mathcal{L}(Y,X)$ onto, such that $\|Q\| = 1$ et $Q \circ f = Id_X$.

Mazur-Ulam, 32 : If f is onto, then f is linear.

Theorem [Kalton-Lancien, 08]. Let M be a separable metric space. Then, there exists a λ-embedding $f : M \to c_0$ with $\lambda = 2$.

Aharoni 74 : $\lambda = 6 + \varepsilon$
Assouad 78 : $\lambda = 3 + \varepsilon$
Pelant 94 : $\lambda = 3$

Optimality : There is no $f : \ell^1 \to c_0$ λ-embedding with $\lambda < 2$.

Theorem [Godefroy-Kalton-Lancien, 00]. If X is a Banach Lipschitz-equivalent to c_0, then X est linearly isomorphic to c_0.

Open problem : Does there exists two separable Banach spaces X and Y Lipschitz equivalent, but not isomorphic ?

Theorem [Prochazka-Sanchez, preprint]. There is a separable metric space M such that, whenever X is a Banach space and $f : M \to X$ is a λ-embedding with $\lambda < 2$, then $\ell^1 \subset X$.

Kalton-Lancien approch for embeddings into c_0.

Definition : (M,d) has property $\Pi(\lambda)$ if,
$\forall \mu > \lambda$, $\exists \nu > \mu$ so that $\forall B_1$, B_2 balls of radii r_1 and r_2, there are subsets $U_1, \cdots, U_N$ and $V_1, \cdots, V_N$ of M, such that

$$\{(x,y) \in B_1 \times B_2;\, d(x,y) > \mu(r_1 + r_2)\} \subset \bigcup_{n=1}^{N} U_n \times V_n$$

and $$\forall n \quad \nu(r_1 + r_2) \leq \lambda \delta(U_n, V_n)).$$

A separable metric space has property $\Pi(2)$.

If a separable metric space has $\Pi(\lambda)$, where $1 < \lambda \leq 2$. Then the exists $f : M \to c_0$ such that for all $x, y \in M$, $x \neq y$, we have :

$$d(x,y) < \|f(x) - f(y)\| \leq \lambda d(x,y).$$

Conversely, if M λ-embeds in c_0, then M has property $\Pi(\mu)$ for all $\mu > \lambda$.

$E \subset M \times M$, $E \neq \emptyset$. $diam(E) = \sup\{d(x,y);\ (x,y) \in E\}$.
$\widetilde{E}$ smallest rectangle containing E. $\delta(E) = \inf\{d(x,y);\ (x,y) \in E\}$.

Definition : (M,d) has property $\pi(\lambda)$ if,
$\forall B_1$, B_2 balls of radii r_1 and r_2, $\forall E \subset B_1 \times B_2$, if $\delta(E) > \lambda(r_1 + r_2)$, then there is a partition $\{E_1, \cdots, E_N\}$ of E so that

$$\forall n, \quad diam(E_n) < \lambda\delta(\widetilde{E}_n)$$

Fact : M has $\Pi(\lambda)$ implies M has $\pi(\lambda)$.

Definition. $f = (f_n) : M \to c_0$ is a strong-λ-embedding if
$\forall x, y \in M, \quad d(x,y) \leq \|f(x) - f(y)\|$, and
$\exists(\lambda_n)$ such that for all n, $\lambda_n < \lambda$ and f_n is λ_n-Lipschitz.

Proposition. If there exists a strong-λ-embedding f from M into c_0, then (M,d) has $\pi(\lambda)$.

Fact : *Let* $E \subset M \times M$ *be bounded,* F *finite dimensional vector normed space,* $P : M \to F$ *be* μ*-Lipschitz continuous such that* $d(x,y) \leq \|P(x) - P(y)\|$ *for each* $(x,y) \in E$*, and* $\varepsilon > 0$*. There exists a partition* $\{E_1, \cdots, E_N\}$ *of* E *so that for each* n*,* $diam(E_n) < \mu\delta(\widetilde{E}_n) + \varepsilon$*.*

Assume $f : M \to c_0$ *strong-*λ*-embedding.*
$B_1 = B(a_1, r_1)$ *and* $B_2 = B(a_2, r_2)$ *balls in* M*.*
$E \subset B_1 \times B_2$ *such that* $\delta(E) > \lambda(r_1 + r_2)$*.*

$$f(x) = \sum_{k=1}^{i_0} f_k(x)e_k + \sum_{k=i_0+1}^{+\infty} f_k(x)e_k = P(x) + Q(x),$$

where i_0 *such that* $\|Q(a_1) - Q(a_2)\| < \delta(E) - \lambda(r_1 + r_2)$*.*
If $(x,y) \in E$*, then* $\|Q(x) - Q(y)\| < \delta(E) \leq d(x,y) \leq \|f(x) - f(y)\|$
Hence, $\|f(x) - f(y)\| = \|P(x) - P(y)\| \geq d(x,y)$ *and* P *is* μ*-Lipschitz with* $\mu < \lambda$*.*

Using the fact, for $\varepsilon > 0$*, there exists a partition* $\{E_1, \cdots, E_N\}$ *of* E *such that for each* n*,* $diam(E_n) < \mu\delta(\widetilde{E}_n) + \varepsilon < \lambda\delta(\widetilde{E}_n)$*.*

Example 1 : *If the bounded subsets of M are totally bounded, then M has property $\pi(1+\varepsilon)$ for all $\varepsilon > 0$.*

Example 2 : *If (M,d) is a separable metric space, then (M,d) has property $\pi(2)$.*

Example 3 : *ℓ^p has property $\pi(2^{1/p})$.*

Theorem : *Let (M,d) be a separable metric space and $1 < \lambda \leq 2$. The following conditions are equivalent :*

(1) (M,d) has property $\pi(\lambda)$.

(2) (M,d) strongly-λ-embeds into c_0.

(3) There exists a strong-λ-embedding $f = (f_n) : M \to c_0$ such that, for all $x, y \in M$, $x \neq y$,

$$d(x,y) < \|f(x) - f(y)\| \leq \lambda d(x,y).$$

Theorem : *Let (M,d) be a separable metric space and $1<\lambda\leq 2$. (M,d) has property $\pi(\lambda)$ if and only if M strongly-λ-embeds into c_0 and,*

$$(*) \quad \forall x,y\in M, x\neq y \quad \Rightarrow \quad d(x,y)<\|f(x)-f(y)\|\leq \lambda d(x,y).$$

Corollary. Let M be a separable metric space. Then, there exists $f : M\to c_0$ satisfying $(*)$ with $\lambda=2$.

Corollary. Let M be a separable metric space such that its bounded subsets are totally bounded. Then there exists a $(1+\varepsilon)$-embedding $f : M\to c_0$.

Corollary. There exists $f : \ell^p\to c_0$ satisfying $(*)$ with $\lambda=2^{1/p}$.

These results are optimal :

The unit sphere of ℓ^p does not λ-embeds into c_0 whenever $\lambda<2^{1/p}$.

Notations : If $x \in M$ and $U, V \subset M$, $d(x,U) := \delta(\{x\} \times U)$,
$diam(U) := diam(U \times U)$, $\delta(U,V) := \delta(U \times V)$.

Lemma 1 : (M,d) *metric,* $U, V, F \subset M$*, non empty and* $\varepsilon \geq 0$.
There exists $f : M \to \mathbb{R}$*, 1-Lipschitz, such that :*
1) For all $x \in F$, $|f(x)| \leq \varepsilon$,
2) For all $(x,y) \in U \times V$,

$$f(x) - f(y) = \min\Big\{\delta(U,V), \delta(U,F) + \delta(V,F) + 2\varepsilon\Big\}.$$

Lemma 2 : *If* (M,d) *has property* $\pi(\lambda)$ *with* $1 < \lambda \leq 2$, $F \subset G$ *are finite subsets of* M *and* $0 < \alpha < \beta$*, we set :*

$$A(F,\beta) = \Big\{(x,y) \in M \times M;\, d(x,y) \geq \lambda\Big(d(x,F) + d(y,F) + \beta\Big)\Big\}$$

Then there is a partition $\{E_1, \cdots, E_N\}$ *of* $A(G,\alpha) \backslash A(F,\beta)$*, such that, for all* n*, if we denote* $\widetilde{E}_n = U_n \times V_n$,

$$diam(E_n) < \lambda \min\Big\{\delta(U_n,V_n), \delta(U_n,F) + \delta(V_n,F) + 2\beta\Big\}.$$

Lemma 1 : *(M, d) metric, $U, V, F \subset M$, non empty and $\varepsilon \geq 0$.*
There exists $f : M \to \mathbb{R}$, 1-Lipschitz, such that :
1) For all $x \in F$, $\quad |f(x)| \leq \varepsilon$,
2) For all $(x, y) \in U \times V$,

$$f(x) - f(y) = \min\Big\{\delta(U, V), \delta(U, F) + \delta(V, F) + 2\varepsilon\Big\}.$$

Fix s, t such that $-\delta(V, F) - \varepsilon \leq s \leq 0 \leq t \leq \delta(U, F) + \varepsilon$ and $t - s = \min\Big\{\delta(U, V), \delta(U, F) + \delta(V, F) + 2\varepsilon\Big\}$. Define

$$f(x) := \min\Big\{d(x, U) + t, d(x, V) + s, d(x, F) + \varepsilon\Big\}$$

The function f is 1-Lipschitz.

If $x \in U$, $f(x) = t$. If $y \in V$, $f(y) = s$.
So, if $x \in U$ and $y \in V$, then $f(x) - f(y) = t - s$.

Finally, if $x \in F$, then $|f(x)| \leq \varepsilon$.

Lemma 2 : *If (M,d) has property $\pi(\lambda)$ with $1 < \lambda \leq 2$, $F \subset G$ are finite subsets of M and $0 < \alpha < \beta$, we set :*

$$A(F,\beta) = \Big\{(x,y) \in M \times M;\, d(x,y) \geq \lambda\Big(d(x,F) + d(y,F) + \beta\Big)\Big\}$$

Then there is a partition $\{E_1, \cdots, E_N\}$ of $A(G,\alpha) \backslash A(F,\beta)$, such that, for all n, if we denote $\widetilde{E}_n = U_n \times V_n$,

$$diam(E_n) < \lambda \min\Big\{\delta(U_n, V_n), \delta(U_n, F) + \delta(V_n, F) + 2\beta\Big\}.$$

If $\Delta = A(G,\alpha) \backslash A(F,\beta)$, $\exists B \subset M$ bounded so that $\Delta \subset B \times B$, because if $(x,y) \in A(G,\alpha)$, then

$$\lambda\Big(d(x,G) + d(y,G)\Big) \leq d(x,y) \leq d(x,G) + d(y,G) + diam(G).$$

There is a partition $\{B_1, B_2, \cdots, B_m\}$ of B such that $\forall j$, $\forall x, x' \in B_j$, $\forall a \in G$, $\quad |d(x,a) - d(x',a)| \leq \alpha/4$.

G finite $\Rightarrow \exists a_j \in G$, $d(B_j, a_j) = \delta(B_j, G)$, so $B_j \subset \overline{B}(a_j, r_j)$, where $r_j = \delta(B_j, G) + \alpha/4$.

The $E_{jk} = E \cap B_j \times B_k$ *form a partition of* Δ, $E_{jk} \subset \overline{B}(a_j, r_j) \times \overline{B}(a_k, r_k)$. *If* $(x, y) \in E_{jk}$:

$$d(x,y) \geq \lambda\Big(d(x,G) + d(y,G) + \alpha\Big) \geq \lambda\Big(\delta(B_j,G) + \delta(B_k,G) + \alpha\Big).$$

So

$$\delta(E_{jk}) > \lambda(r_j + r_k).$$

Applying $\pi(\lambda)$ *to each* E_{jk}, *there is a partition* $\{E_1, \cdots, E_N\}$ *of* Δ *such that,*

$$diam(E_n) < \lambda\delta(\widetilde{E}_n) = \lambda\delta(U_n, V_n), \quad \text{where } \widetilde{E}_n = U_n \times V_n$$

Moreover, if $E_n \subset B_j \times B_k$ *and* $(x, y) \in E_n$, *then*

$$d(x,y) \leq \lambda\Big(d(x,F) + d(y,F) + \beta\Big) \leq \lambda\Big(\delta(B_j,F) + \delta(B_k,F) + \alpha/2 + \beta\Big)$$

hence

$$diam(E_n) < \lambda\Big(\delta(U_n,F) + \delta(V_n,F) + 2\beta\Big).$$

Proof of Theorem 1.

Goal : find $f_n : M \to \mathbb{R}$ 1-Lipschitz with $(f_n) \to 0$ pointwise, and a partition $\{E_n; n \in \mathbb{N}\}$ of $\{(x,y) \in M \times M; x \neq y\}$, so that for each n, $f_n(x) - f_n(y) = c_n$ if $(x,y) \in E_n$ and $diam(E_n) < \lambda c_n$.

If $\lambda_n < \lambda$ and $diam(E_n) < \lambda_n c_n$, $f = (\lambda_n f_n)$ is a strong-λ-embedding.

Let (a_k) be a dense sequence in M, $F_k = \{a_1, \cdots, a_k\}$, and $(\varepsilon_k) \searrow 0$.
We set $\Delta_k = A(F_k, \varepsilon_k) \backslash A(F_{k+1}, \varepsilon_{k+1})$.
By lemma 2, $\exists 0 = n_1 < n_2 < \cdots < n_k < \cdots$, $\exists E_n \subset M \times M$ such that for all k, $\bigcup_{n_k < n \leq n_{k+1}} E_n = \Delta_k$, and, if $n_k < n \leq n_{k+1}$,

$$diam(E_n) < \lambda \min \Big\{ \delta(U_n, V_n), \delta(U_n, F_k) + \delta(V_n, F_k) + 2\varepsilon_k \Big\}.$$

By lemma 1, $\exists f_n : M \to \mathbb{R}$, 1-Lipschitz, such that
1) if $x \in F_k$ and $n_k < n \leq n_{k+1}$, then $|f_n(x)| \leq \varepsilon_k$,
2) if $n_k < n \leq n_{k+1}$ and $(x,y) \in U_n \times V_n$, then

$$f_n(x) - f_n(y) = c_n := \min \Big\{ \delta(U_n, V_n), \delta(U_n, F_k) + \delta(V_n, F_k) + 2\varepsilon_k \Big\}.$$

So $diam(E_n) < \lambda c_n$.

The $E_n's$ form a partition of $\{(x,y) \in M \times M;\, x \neq y\}$.
Indeed, if $x, y \in M$, $x \neq y$ and if $\sigma_k = \lambda\Big(d(x, F_k) + d(y, F_k) + \varepsilon_k\Big)$,
then $0 < d(x,y) < \sigma_1$, $(\sigma_k) \searrow 0$,
so $\exists k$ such that $\sigma_{k+1} \leq d(x,y) < \sigma_k$, which means $(x,y) \in \Delta_k$,
thus, $\exists n$ such that $(x,y) \in E_n$.

If $x \in M$, $\lim_{n \to \infty} f_n(x) = 0$.
It is true if $x = a_j$ for some j.
The f_n's are 1-Lipschitz and (a_n) is dense, it is true for all x.

Remark : A metric space strongly λ-embeds into c_0 if and only if its bounded subsets strongly λ-embed into c_0.

Example : There exists $M \subset \ell^1$ such that
M does not 2-embed into c_0^+
although its bounded subsets $(1+\varepsilon)$-embed int c_0
and M $(1+\varepsilon)$-embeds into c_0 for all $\varepsilon > 0$.

Construction : $a_0 = e_0/2$, $a_1 = 0$, $a_2 = e_0$, and for $r \in \mathbb{N}$, $F_r = \{a_1 + re_n, a_2 + re_n;\, n \leq r\}$.

$$M = \{a_0\} \cup \bigcup_{n\in\mathbb{N}} F_r$$