Smoothness of quasihyperbolic balls in convex domains of Banach spaces

Conference on Geometric Functional Analysis and its Applications

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[1] Joint work with R, Klén, A. Rasila

Abstract

Quasihyperbolic (QH) metric is a weighted metric on a path-connected metric space motivated by looking at invariants of Möbius tranformations of the unit disk. In this talk the metric is given on an open convex non-trivial subset of a Banach space. It turns out that many properties of the underlying Banach space are visible in the QH geometry.

We address the following problems here: when are QH-metric balls C^1-smooth? It turns out that the answer is affirmative for uniformly smooth Banach spaces.

This problem was considered by F.W Gehring and M. Vuorinen in the '70s. To the best of our knowledge it was open until now even in $\mathbb{R}^n$. These consideration also lead to a renorming technique of Banach spaces.

Hyperbolic metric in the unit disk

Suppose that $w = f(z)$ is a conformal mapping of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ onto itself. Then by Pick's lemma we have the equality

$$\left|\frac{dw}{dz}\right| = \frac{1 - |w|^2}{1 - |z|^2}.$$

This identity can be written as

$$\frac{|dw|}{1 - |w|^2} = \frac{|dz|}{1 - |z|^2},$$

which means that for any regular curve γ in the unit disk, and for any conformal self mapping f of $\mathbb{D}$, we have

$$\int_{f\circ\gamma} \frac{|dw|}{1 - |w|^2} = \int_{\gamma} \frac{|dz|}{1 - |z|^2}.$$

We have obtained a length function which is invariant under conformal mappings of the unit disk onto itself.

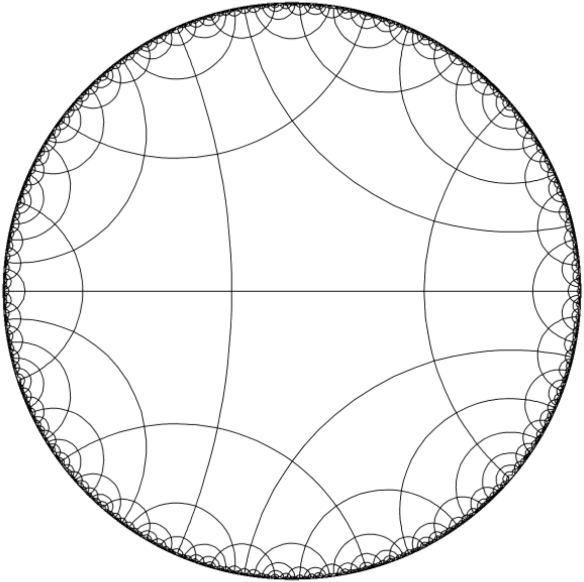

Figure : Hyperbolic Geodesics in Poincaré disc

Quasihyperbolic metric

Let X be a Banach space and $\Omega \subset \mathrm{X}$ a domain. By a domain we mean in this context an open path-connected proper subset of X.

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For $x \in \Omega$, let $d(x) = d_{\partial\Omega}(x)$ denote the distance $\mathrm{dist}(x, \partial\Omega)$. We define the *quasihyperbolic length* of γ by

$$\ell_k(\gamma) := \int \frac{\|\gamma'(t)\|}{d_{\partial\Omega}(\gamma(t))}\, dt$$

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Then the *quasihyperbolic distance* of points $x, y \in \Omega$ is the number

$$k_\Omega(x, y) := \inf_{\gamma} \ell_k(\gamma)$$

where the infimum is taken over all Lipschitz paths γ joining x and y in Ω.

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Some related references

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J. Väisälä: Quasihyperbolic geometry of domains in Hilbert spaces, *Ann. Acad. Sci. Fenn. Math.* **32** (2007), 559-578.

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The history of Banach spaces & QH metric

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- (Free does not refer to a universal property but the fact that estimates do not depend on n. Thus Hilbert spaces are on the reach.)
- Therefore some geometric style tools were needed. Quasihyperbolic metrics and the moduli of curve families are the most common of these.

Discussions on the smoothness of QH balls

- At the time of the origin of the QH-ball smoothness problem some mathematicians conjectured that QH balls in a convex region of $\mathbb{R}^n$ need not be smooth.

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$$\Omega = \{(x,y) \in \mathbb{R}^2 \colon -1 < x < 1\}$$

the corresponding QH ball $B_k((0,0),1)$ looks like a rugby ball.

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- The first guess was that the tip of the ball is pointed.
- This particular suggestion was later refuted by M. Vuorinen by lengthy calculations.

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Theorem B

Let X be a uniformly smooth Banach space and let $\Omega \subset \mathrm{X}$ be a convex domain. Then the quasihyperbolic balls are smooth in the sense that $k(x_0, \cdot)$, $x_0 \in \Omega$, is continuously Fréchet differentiable away from x_0.

Some general remarks

- A QH *geodesic* is a Lipschitz path $\gamma\colon [0,1] \to \Omega$ such that it has the minimal QH-length, $\ell_k(\gamma)$, among the paths λ joining the end points $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$.

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- By virtue of the RNP and suitable normalizations of γ we may assume that γ' exists a.e. and we may recover γ by integrating γ' in the Bochner sense,

$$\gamma(t) = \gamma(0) + \int_0^t \gamma'(s)\, ds.$$

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- Moreover, if the space is uniformly convex of some power type, then the geodesics are C^1-smooth.
- Any simple-minded argument must use the notions readily available (e.g. geodesics).
- On the other hand, he geodesics are quite handy to work with.

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- Suppose that x is a point on a QH sphere and z, y are points close by. Then we would expect the geodesics γ_x (between x_0, x), γ_z (between x_0, z) and γ_y (between x_0, y) to be running pretty close.

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- Therefore, we require paths λ_z and λ_y which are close to γ_x at an early stage (to avoid errors accumulating) but which must depart from it well before the end.
- This is the rough idea. The suitable form can be 'reverse-engineered' by looking at the type of calculations that follow.

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Sketch of Pf. of Thm. B

- We wish to show that

$$\frac{k(\gamma(0),\gamma(\ell)+h)+k(\gamma(0),\gamma(\ell)-h)-2k(\gamma(0),\gamma(\ell))}{\|h\|}\to 0 \qquad (1)$$

as $\|h\|\to 0$ uniformly, not depending on the particular endpoints or norm-length ℓ, but with ℓ uniformly bounded away from zero.

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- This yields uniform Fréchet differentiability of $k_{x_0}(\cdot)$ and from it the C^1-smoothness can be obtained.
- Fix $h\in \mathrm{X}$ such that $\gamma(\ell)\pm h\in\Omega$. Write $t=\sqrt{\|h\|}$.
- Define new paths γ_+ and γ_- as follows: on the segment $[0,\ell-t]$ they coincide with γ and

$$\gamma_\pm(\ell-t+s)=\gamma(\ell-t+s)\pm s^2\frac{h}{\|h\|}\ \text{for } 0\leq s\leq t.$$

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- Note that
$$\|(\gamma_\pm(\ell - t + s) - \gamma(\ell - t + s))'\| = \left\| \pm \frac{d}{ds} s^2 \frac{h}{\|h\|} \right\| = 2s.$$

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- By a mean value principle we obtain

$$\frac{1}{d(\gamma_\pm(\ell - t + s), \partial\Omega)} \leq \frac{1}{d(\gamma(\ell - t + s), \partial\Omega)} + s^2 d^*.$$

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- Putting pieces together...

$$
\begin{aligned}
k(\gamma(0),\gamma(\ell)+h) + k(\gamma(0),\gamma(\ell)-h) - 2k(\gamma(0),\gamma(\ell))& \\
\leq \ell_k(\gamma_+) + \ell_k(\gamma_-) - 2\ell_k(\gamma)& \\
= \int_{s=0}^{t} \frac{\|\gamma_+'(\ell-t+s)\|}{d(\gamma_+(\ell-t+s),\partial\Omega)}\, ds + \int_{s=0}^{t} \frac{\|\gamma_-'(\ell-t+s)\|}{d(\gamma_-(\ell-t+s),\partial\Omega)}\, ds& \\
- 2\int_0^t \frac{ds}{d(\gamma(\ell-t+s),\partial\Omega)}& \\
\leq \int_{s=0}^{t} 2(1+\mu_{\|\cdot\|}(2s)) \left(\frac{1}{d(\gamma(\ell-t+s),\partial\Omega)} + s^2 d^* \right) ds& \\
- 2\int_0^t \frac{ds}{d(\gamma(\ell-t+s),\partial\Omega)}&.
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\begin{aligned}
&k(\gamma(0),\gamma(\ell)+h)+k(\gamma(0),\gamma(\ell)-h)-2k(\gamma(0),\gamma(\ell))\\
&\qquad\qquad\leq \ell_k(\gamma_+)+\ell_k(\gamma_-)-2\ell_k(\gamma)\\
&=\int_{s=0}^{t}\frac{\|\gamma_+'(\ell-t+s)\|}{d(\gamma_+(\ell-t+s),\partial\Omega)}\,ds+\int_{s=0}^{t}\frac{\|\gamma_-'(\ell-t+s)\|}{d(\gamma_-(\ell-t+s),\partial\Omega)}\,ds\\
&\qquad\qquad-2\int_0^t\frac{ds}{d(\gamma(\ell-t+s),\partial\Omega)}\\
&\leq\int_{s=0}^{t}2(1+\mu_{\|\cdot\|}(2s))\left(\frac{1}{d(\gamma(\ell-t+s),\partial\Omega)}+s^2d^*\right)ds\\
&\qquad\qquad-2\int_0^t\frac{ds}{d(\gamma(\ell-t+s),\partial\Omega)}.
\end{aligned}
$$

- We claim that the bottom value converges to 0 faster than $\|h\|=t^2\to 0$. Since $s^2\leq t^2\to 0$ we are only required to investigate the term involving $\mu_{\|\cdot\|}$.

- Since $y \mapsto \frac{1}{d(y,\partial\Omega)}$ is Lipschitz on B an easy argument gives that there is small enough $\|h\|$ such that

$$1/d(\gamma(\ell - t + s), \partial\Omega) \leq 2/d(\gamma(\ell), \partial\Omega) \text{ for } s \leq t.$$

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- Let us estimate

$$\begin{aligned}
&\frac{1}{t^2}\int_0^t 2\mu_{\|\cdot\|}(2s)\left(\frac{1}{d(\gamma(\ell - t + s), \partial\Omega)} + s^2 d^*\right) ds \\
&\qquad \leq \frac{1}{t^2}\int_0^t 4s\epsilon(2s)\left(\frac{2}{d(\gamma(\ell), \partial\Omega)} + s^2 d^*\right) ds \\
&\qquad \leq \frac{1}{t^2}\Big|_{s=0}^{t} \epsilon(2t)\left(\frac{4s^2}{d(\gamma(\ell), \partial\Omega)} + s^4 d^*\right) \\
&\qquad = \epsilon(2t)\left(\frac{4}{d(\gamma(\ell), \partial\Omega)} + t^2 d^*\right) \to 0, \quad t \to 0.
\end{aligned}$$

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- Reviewing the above calculations yields that if $p, r \in [1, 2)$, $r \leq (p+1)/2$ and $\mu_{\|\cdot\|}(\tau) = \tau^p \epsilon(\tau)$ where $\epsilon(\tau) \searrow 0$ as $\tau \to 0$, then we have

$$\frac{k(x_0, x+h) + k(x_0, x-h) - 2k(x_0, x)}{\|h\|^r} \to 0$$

as $\|h\| \to 0$ uniformly in any annulus whose distance to x_0 and $\partial\Omega$ is strictly positive.

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- This suggests the following considerations...

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Reversing the direction of deductions

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- Interestingly, the quasihyperbolic balls $B_k(0, r)$ in a centered symmetric convex domains of Banach spaces are symmetric and convex as well.
- By applying the Minkowski functional on these bodies one can define an equivalent norm on the space.
- Moreover, the QH-ball-induced norms enjoy some properties of the original norm.
- It is easy to see that when the QH-radius $r \to \infty$, the corresponding QH spheres 'approximate' the boundary of the bounded convex domain uniformly.

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Quasihyperbolic metric as a renorming technique

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- This approach is not sensitive to the possible non-separability of the space.
- The definition of the QH metric appears to 'convolute' norms in a sense. Therefore, it will not cause additional problems itself..?

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Theorem C

Suppose that $(X, \|\cdot\|)$ is a uniformly smooth Banach space. Then each equivalent norm can be approximated uniformly on bounded sets by uniformly smooth norms induced by the quasihyperbolic balls. Moreover, if the modulus of smoothness of the norm $\|\cdot\|$ has power type $1 < p \leq 2$, then the approximating norms have power types $< (p+1)/2$.

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Moreover, if the modulus of smoothness of the norm $\|\cdot\|$ has power type $1 < p \leq 2$, then the approximating norms have power types $< (p+1)/2$.

Theorem D

Suppose that X is a uniformly convex Banach space, $\Omega \subset \mathrm{X}$ is a symmetric convex domain and $B = B_k(0, r)$ for some $r > 0$. Then the equivalent norm $|||\cdot||| = M_B(\cdot)$ on X is uniformly convex.
Moreover, any equivalent norm on X can be approximated uniformly on bounded sets by uniformly convex norms arising from such quasihyperbolic balls B.

- Sketch of proof of Thm. D. Note that

$$k\left(0,\frac{x_n+y_n}{2}\right)\leq\int_0^R\frac{\frac{1}{2}\|(\gamma_n(t)+\lambda_n(t))'\|}{d(\frac{1}{2}(\gamma_n(t)+\lambda_n(t)),\partial\Omega)}\,dt$$
$$\leq\int_0^R\frac{\|v_n(t)+w_n(t)\|}{2}\,dt,$$

where

$$v_n(t):=\frac{\gamma_n'(t)}{d(\gamma_n(t),\partial\Omega)}\text{ and }w_n(t):=\frac{\lambda_n'(t)}{d(\lambda_n(t),\partial\Omega)}$$

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- Sketch of proof of Thm. D. Note that

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and we use the convexity of the domain Ω.

- Here $R > 0$ is the QH radius of the QH sphere. The paths have constant QH-speed, thus $\|v_n(t)\| = \|w_n(t)\| = 1$ for a.e. $t \in [0, R]$.

- Put

$$\mathrm{Aver}_n = \frac{1}{R}\int_0^R \|v_n(t) - w_n(t)\|\ dt.$$

and observe that

$$\int_0^R \frac{\|v_n(t) + w_n(t)\|}{2}\ dt \leq \int_0^R 1 - \hat{\delta}_{\|\cdot\|}(\|v_n(t) - w_n(t)\|)\ dt$$
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$$\leq \int_0^R 1 - \hat{\delta}_{\|\cdot\|}(\mathrm{Aver}_n)\ dt = R - R\hat{\delta}_{\|\cdot\|}(\mathrm{Aver}_n).$$
- Thus, if $k(0,(x_n + y_n)/2) \to R$, then $\mathrm{Aver}_n \to 0$.

Lemma

Suppose that X is a Banach space having the RNP and $w\colon \mathrm{X} \to [a,\infty)$, $a > 0$, is a Lipschitz continuous 'weight' and γ, λ are Lipschitz paths starting from the same point. (Plus some technical assumptions that can be met.) If

$$\gamma'(t)w(\gamma(t)) = \lambda'(t)w(\lambda(t))$$

a.e. then $\gamma = \lambda$.

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- What we really need is a much stronger asymptotic version of the above fact. Namely that if $\int \|v_n(t) - w_n(t)\|\, dt \to 0$ then we should have $\|\gamma_n(R) - \lambda_n(R)\| \to 0$.

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Lemma

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- What we really need is a much stronger asymptotic version of the above fact. Namely that if $\int \|v_n(t) - w_n(t)\| \, dt \to 0$ then we should have $\|\gamma_n(R) - \lambda_n(R)\| \to 0$.
- The bad news is that the argument for the above Lemma is not even stable under mild perturbations of λ and γ.

The fix

- We reduce our considerations to an ultrapower $\mathrm{X}^{\mathcal{U}}$. Since X is superreflexive, $\mathrm{X}^{\mathcal{U}}$ is reflexive, in particular it has the RNP.

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$$\gamma(t) := (\gamma_n(t))^{\mathcal{U}}, \quad \lambda(t) := (\lambda_n(t))^{\mathcal{U}} \colon [0, R] \to \mathrm{X}^{\mathcal{U}}.$$
- The aim is to push the analysis of X to the ultrapower. This appears hopeless at first glance because some uniformity is usually required when working with ultrapowers.

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- Luckily, we have shown previously that QH geodesics α enjoy a *uniformity condition* involving the convergence of the derivative,

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- Then we apply the above Lemma in $\mathrm{X}^{\mathcal{U}}$ and 'pull back' the conclusion to X.
- After running a counter assumption and choosing a filter basis of $\mathcal{U}$ suitably this yields the required statement that

$$\|x_n - y_n\| = \|\lambda_n(R) - \gamma_n(R)\| \to 0, \quad n \to \infty.$$

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Thank you!

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