On an isomorphic Banach-Mazur rotation problem and maximal norms in Banach spaces

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(joint work with Stephen Dilworth)

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Conference on Geometric Functional Analysis and its Applications
Besançon, France
October 27-31, 2014

Banach-Mazur Rotation Problem 1932

Problem

Is every separable Banach space with a transitive group of isometries isometrically isomorphic to a Hilbert space?

Definition

A norm on X is called *transitive* if the orbit under the group of isometries of any element x in the unit sphere of X is equal to the unit sphere of X.
i.e. $\forall x, y \in S_X, \exists T \in \text{Isom}(X) : Tx = y$.

Throughout this talk 'isometry' will mean 'linear surjective isometry'.

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There exist non-separable transitive Banach spaces which are not isomorphic to a Hilbert space.

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Theorem (Pełczyński-Rolewicz)

There exist non-separable transitive Banach spaces which are not isomorphic to a Hilbert space.

Theorem (Lusky 1979)

Every separable Banach space $(X, \|\cdot\|)$ is complemented in a separable almost transitive space $(Y, \|\cdot\|)$.

Definition

A norm on X is called *almost transitive* if the orbit under the group of isometries of any element x in the unit sphere of X is norm dense in the unit sphere of X.
i.e. $\forall x, y \in S_X, \forall \varepsilon > 0 \ \ \exists T \in \text{Isom}(X) : \|Tx - y\| < \varepsilon$.

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The study of isometry groups of renormings of X is equivalent to the study of bounded subgroups of the group $GL(X)$ of all isomorphisms from X onto X. Indeed, if G is a bounded subgroup of $GL(X)$ (i.e. $sup_{g \in G} \|g\| < \infty$) we can define an equivalent norm $\| \cdot \|_G$ on X by

$$\|x\|_G = \sup_{g \in G} \|gx\|.$$

Then G is a subgroup of $\mathrm{Isom}(X, \| \cdot \|_G)$, and $\mathrm{Isom}(X, \| \cdot \|_G)$ is a bounded subgroup of $GL(X)$.

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The spaces ℓ_p, $1 < p < \infty, p \neq 2$,
and all infinite-dimensional subspaces of their quotient spaces do not admit equivalent almost transitive renormings.

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Key Lemma

Suppose that a Banach space X *with a Schauder basis contains an almost transitive subspace* Y.
Then, given $\delta > 0$, $y_0 \in Y$, *and* $N \in \mathbb{N}$,
there exists $y \in Y$, *with* $\|P_N^X(y)\| < \delta$, *such that for all scalars* a, b, *we have*

$$(1-\delta)\Big(|a|^2\|y_0\|^2+|b|^2\Big)^{\frac{1}{2}} \leq \|ay_0 + by\| \leq (1+\delta)\Big(|a|^2\|y_0\|^2+|b|^2\Big)^{\frac{1}{2}}.$$

Proof of Key Lemma

By compactness $\exists n := n(N, \delta)$ such that if $(y_i)_{i=1}^n \subset B_Y$ then $\exists 1 \leq i < j \leq n$ such that $\|P_N^X(y_j - y_i)\| < \delta$.

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By Dvoretzky's theorem and almost transitivity of Y, $\exists (y_i)_{i=1}^n \subset S_Y$ such that for all scalars $a, b_1, \ldots, b_n$, we have

$$\begin{aligned}(1-\delta)\left(|a|^2\|y_0\|^2 + \sum_{i=1}^n |b_i|^2\right)^{1/2} &\leq \left\|ay_0 + \sum_{i=1}^n b_i y_i\right\| \\ &\leq (1+\delta)\left(|a|^2\|y_0\|^2 + \sum_{i=1}^n |b_i|^2\right)^{1/2}\end{aligned}$$

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(1-\delta)\left(|a|^2\|y_0\|^2 + \sum_{i=1}^n |b_i|^2\right)^{1/2} &\leq \left\| a y_0 + \sum_{i=1}^n b_i y_i \right\| \\
&\leq (1+\delta)\left(|a|^2\|y_0\|^2 + \sum_{i=1}^n |b_i|^2\right)^{1/2}
\end{aligned}
$$

Choose $1 \leq i < j \leq n$ such that $\|P_N^X(y_j - y_i)\| < \delta$ and set $y = (1/\sqrt{2})(y_j - y_i)$.

Theorem

Suppose that a Banach space X with a Schauder basis contains an almost transitive subspace Y.
Let $r \in FR(Y) := \{1 \leq r \leq \infty \colon \ell_r$ is finitely representable in $Y\}$.

Then, given $\varepsilon > 0$ and any sequence (a_i) of nonzero scalars, there exists a normalized block basis (x_i) in X such that, for all $m \geq 1$ and all scalars b, we have

$$(1-\varepsilon)\left(\sum_{k=1}^{m}|a_k|^r + |b|^r\right)^{1/r} \leq \left\|\sum_{k=1}^{m} a_k x_k + b x_{m+1}\right\| \leq (1+\varepsilon)\left(\sum_{k=1}^{m}|a_k|^r + |b|^r\right)^{1/r}.$$

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Proof:

By previous theorem for all $n \in \mathbb{N}$ and any $\varepsilon > 0$ there exists a normalized block basis $(x_k)_{k=1}^n$ in X such that,

$$(1-\varepsilon)n^{1/2} \leq \Big\| \sum_{k=1}^{n} x_k \Big\| \leq (1+\varepsilon)n^{1/2}.$$

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But in $X = \ell_p$, $1 \leq p < \infty$, every block basis is isometrically equivalent to the standard basis of ℓ_p,

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Recall:
X is an asymptotic-ℓ_p space if there exists $C > 0$ such that for all $n \leq x_1 < \cdots < x_n$, we have

$$\frac{1}{C}\left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p} \leq \left\|\sum_{k=1}^{n} x_k\right\| \leq C\left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p}.$$

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In the previous theorem we can start the block basis after n and we get that for all $n \in \mathbb{N}$ and any $\varepsilon > 0$ there exists a normalized block basis $n < (x_k)_{k=1}^n$ in X such that,

$$(1-\varepsilon)n^{1/2} \leq \left\|\sum_{k=1}^{n} x_k\right\| \leq (1+\varepsilon)n^{1/2}.$$

and the resulting contradiction proves the following corollary.

Corollary

No subspace of any asymptotic-ℓ_p *space* , $1 \leq p < \infty$, $p \neq 2$, *admits an almost transitive renorming.*

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For $1 < p < \infty$, let $\mathcal{C}_p$ denote the class of Banach spaces which are isomorphic to a subspace of an ℓ_p-sum of finite-dimensional normed spaces.

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Let $1 < p < \infty$, $p \neq 2$. If $X \in \mathcal{C}_p$ then X does not admit an equivalent almost transitive norm.

It is known that $\mathcal{C}_p$ contains every infinite-dimensional subspace of a quotient space of ℓ_p [Johnson, Zippin 1974].

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It is known that $\mathcal{C}_p$ contains every infinite-dimensional subspace of a quotient space of ℓ_p [Johnson, Zippin 1974].
Thus no infinite-dimensional subspace of a quotient space of ℓ_p admits an equivalent almost transitive norm.

Recall the notion of asymptotic structure introduced by Maurey, Milman, and Tomczak-Jaegermann (1992):

A basis $(b_i)_{i=1}^n$ of unit vectors for an n-dimensional normed space belongs to $\{X, (e_i)\}_n$ if

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$\forall \varepsilon > 0 \; \forall m_1 \, \exists x_1 > m_1$ s.t. $\forall m_2 \, \exists x_2 > m_2$ s.t. $\ldots \forall m_n \, \exists x_n > m_n$ such that there exist c and C, with $0 < c \leq C$ and $C/c < 1 + \varepsilon$, such that for all scalars $(a_i)_{i=1}^n$, we have

$$c \left\| \sum_{i=1}^n a_i b_i \right\| \leq \left\| \sum_{i=1}^n a_i x_i \right\| \leq C \left\| \sum_{i=1}^n a_i b_i \right\| .$$

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In this language our result states that if X has a Schauder basis (e_i) and contains an almost transitive subspace Y then for all $r \in FR(Y)$, the unit vector basis of ℓ_r^2 belongs to $\{X,(e_i)\}_2$.

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Theorem

Suppose that X has an unconditional basis (e_i). If X contains an almost transitive subspace Y, then there exist $p \in [p_X, p_Y]$ and $q \in [q_Y, q_X]$ such that, for $r = p$ and $r = q$ and for all $n \geq 1$ and $\varepsilon > 0$, there exist disjointly supported vectors $(x_i)_{i=1}^n \subset X$ such that $(x_i)_{i=1}^n$ is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_r^n.
In particular, if $Y = X$, then $p = p_X$ and $q = q_X$.

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Proof relies on a result of Sari (2004) about disjoint envelope functions.

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Question Fix $m \in \mathbb{N}$, $\varepsilon > 0$. Let X have an uncoditional basis such that $(\{x_i\}_{i=1}^m)$ is $(1+\varepsilon)$-equivalent to the standard unit basis of ℓ_r^m, with $r \neq 2$. Can X be transitive?

Recall that a basis satisfies (p,q)-estimates, where $1 < q \le p < \infty$, if there exists $C > 0$ such that

$$\frac{1}{C}\left(\sum_{k=1}^{n}\|x_k\|^p\right)^{1/p} \le \left\|\sum_{k=1}^{n} x_k\right\| \le C\left(\sum_{k=1}^{n}\|x_k\|^q\right)^{1/q},$$

whenever $x_1 < x_2 < \cdots < x_n$.

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Corollary

Suppose that a Banach space X with a Schauder basis (e_i) contains a subspace Y which admits an equivalent almost transitive norm. If (e_i) satisfies (p, q)-estimates, then $q \leq 2 \leq p$.

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Corollary

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Proof.

Let $\||\cdot\||$ be the equivalent almost transitive norm on Y. Then $\||\cdot\||$ extends to an equivalent norm $\||\cdot\||$ on X.
Thus the Main Theorem gives the desired conclusion. □

Corollary

Suppose that X is an Orlicz sequence space ℓ_M (where M is an Orlicz function). Then X contains a subspace Y which admits an almost transitive norm if and only if X contains a subspace isomorphic to ℓ_2.

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Proof.

Previous Corollary implies that the Matuszewska-Orlicz indices of M satisfy $\alpha_M \leq 2 \leq \beta_M$, which in turn implies by a theorem of Lindenstrauss and Tzafriri that ℓ_M contains a subspace isomorphic to ℓ_2.

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Conversely, if X contains a subspace Y isomorphic to ℓ_2 then Y admits an equivalent transitive norm. □

We combine our results with known results about the structure of subspaces of L_p, $2 < p < \infty$, and we obtain that if X is a subspace of L_p, $2 < p < \infty$, such that every subspace of X admits an equivalent almost transitive norm, then X is isomorphic to a Hilbert space (this also is true in the non-commutative case).

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This suggests the following question.

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Suppose that every subspace of a Banach space X admits an equivalent almost transitive (or transitive) renorming. Is X isomorphic to a Hilbert space?

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YES, if such a space X is a stable and admits a C^2-bump.

Maximal and non-maximal norms

Definition (Pełczyński and Rolewicz 1962)

A Banach space $(X, \|\cdot\|)$ is called maximal if whenever $|||\cdot|||$ is an equivalent norm on X so that $\mathrm{Isom}(X, \|\cdot\|) \subseteq \mathrm{Isom}(X, |||\cdot|||)$, then $\mathrm{Isom}(X, \|\cdot\|) = \mathrm{Isom}(X, |||\cdot|||)$.

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A 'maximal isometry group of a renorming of X' is the same as a 'maximal bounded subgroup of $GL(X)$' (where $GL(X)$ denotes the group of linear surjective isomorphisms on X).

Any bounded subgroup G of $GL(X)$ is a subgroup of the isometry group of $(X, |||.|||)$ where

$$|||x||| = \sup_{g \in G} \|gx\|.$$

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In 1982 Wood posed a problem whether every Banach space admits an equivalent maximal norm, that is whether $GL(X)$ always has maximal bounded subgroups.

In 2013 Ferenczi and Rosendal answered this problem negatively by constructing a complex super-reflexive space and a real reflexive space, both without an equivalent maximal renorming.

In 2006 Wood asked, what he called a more natural question, whether for every Banach space there exists an equivalent maximal renorming whose isometry group contains the original isometry group,
that is, whether every bounded subgroup of $GL(X)$ is contained in a maximal bounded subgroup of $GL(X)$.

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that is, whether every bounded subgroup of $GL(X)$ is contained in a maximal bounded subgroup of $GL(X)$.

We show that even in Banach spaces which admit an equivalent maximal norm there can also exist equivalent norms with isometry groups which are not contained in any maximal isometry group.

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Consider renormings of the form

$$Z = \left(\sum_{n=1}^{\infty} \oplus \ell_2^{k_n} \right)_{\ell_p}$$

where k_n is a sequence of natural numbers, so that $\forall n \in \mathbb{N}$,

$$\sum_{j=1}^{n-1} k_j < k_n.$$

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Isometries of this kind of spaces were described by H. Rosenthal.

Theorem (Rosenthal 1986)

X – a pure space with a 1-unconditional basis $E = \{e_\gamma\}_{\gamma\in\Gamma}$
(i.e. $\forall\gamma_1 \neq \gamma_2, \mathrm{span}\{e_{\gamma_1}, e_{\gamma_2}\}$ is not cannonically isometric to ℓ_2^2)
$(H_\gamma)_{\gamma\in\Gamma}$ – Hilbert spaces all of dimension at least 2,
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$Z = (\sum_{\gamma\in\Gamma} \oplus H_\gamma)_E$ – is called a functional hilbertian sum.
Let $P(Z)$ be the set of all bijections $\sigma : \Gamma \to \Gamma$ so that*

(a) *$\{e_{\sigma(\gamma)}\}_{\gamma\in\Gamma}$ is isometrically equivalent to $\{e_\gamma\}_{\gamma\in\Gamma}$, and*

(b) *$H_{\sigma(\gamma)}$ is isometric to H_γ for all $\gamma \in \Gamma$.*

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(i.e. $\forall\gamma_1 \neq \gamma_2, \mathrm{span}\{e_{\gamma_1}, e_{\gamma_2}\}$ is not cannonically isometric to ℓ_2^2)
$(H_\gamma)_{\gamma\in\Gamma}$ – Hilbert spaces all of dimension at least 2,
$Z = (\sum_{\gamma\in\Gamma} \oplus H_\gamma)_E$ – is called a functional hilbertian sum.
Let $P(Z)$ be the set of all bijections $\sigma : \Gamma \to \Gamma$ so that

(a) *$\{e_{\sigma(\gamma)}\}_{\gamma\in\Gamma}$ is isometrically equivalent to $\{e_\gamma\}_{\gamma\in\Gamma}$, and*
(b) *$H_{\sigma(\gamma)}$ is isometric to H_γ for all $\gamma \in \Gamma$.*

Then $T : Z \to Z$ is a surjective isometry if and only if $\exists\sigma \in P(Z)$ and surjective isometries $T_\gamma : H_\gamma \to H_{\sigma(\gamma)}$ ($\gamma \in \Gamma$), so that for all $z = (z_\gamma)_{\gamma\in\Gamma}$ in Z, and for all $\gamma \in \Gamma$,

$$(Tz)_{\sigma(\gamma)} = T_\gamma(z_\gamma).$$

Let $p \neq 2$ and

$$Z = \left(\sum_{n=1}^{\infty} \oplus \ell_2^{k_n} \right)_{\ell_p},$$

where $k_n \in \mathbb{N}$ are such that $k_1 \geq 2$ and $\forall n \in \mathbb{N}$, $\sum_{j=1}^{n-1} k_j < k_n$.

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Thus $\forall m \in \mathbb{N}$

$$Z_m = (\ell_2^d) \oplus_p \left(\sum_{n=m}^{\infty} \oplus \ell_2^{k_n} \right)_{\ell_p},$$

where $d = \sum_{j=1}^{m} k_j$, is isomorphic to Z, $P(Z_m)$ also consists of the identity only, and

$$\mathrm{Isom}(Z_m) \subsetneq \mathrm{Isom}(Z_{m+1}).$$

Using the same method we obtain

Theorem

The following spaces have a continuum of renormings none of whose isometry groups is contained in any maximal isometry group of an equivalent renorming.

- *Spaces* ℓ_p, $1 < p < \infty$, $p \neq 2$,

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Question: Does $X = T^{(2)}$, or X a general weak Hilbert space, other than ℓ_2, have a maximal bounded subgroup of $GL(X)$?

Maximal renormings

Theorem

For $1 < p < \infty$, $p \neq 2$, ℓ_p admits a continuum of renormings whose isometry groups are maximal and are not pairwise conjugate in the isomorphism group of ℓ_p.

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Then $P(Z_m)$ consists of all permutations, and the group $\mathrm{Isom}(Z_m)$ is maximal

To get continuum mutually noncojugate renormings
let $J \subset \mathbb{N}$, with $\min J \geq 2$,
and let $\mathbb{N} = \bigcup_{j \in J} A_j$, where A_j are disjoint infinite subsets of $\mathbb{N}$.
For $k \in A_j$, let $H_k = \ell_2^j$, and

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This works for any space

$$Z = (\sum_{k=1}^{\infty} \oplus \ell_2^k)_E,$$

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where $E = \{e_k\}_{k=1}^{\infty}$ is a pure 1-symmetric basis for some X which is not isomorphic to ℓ_2.
It is well-known that if $(k_j)_{j=1}^{\infty} \subset \mathbb{N}$ is any unbounded sequence then $(\sum_{j=1}^{\infty} \oplus \ell_2^{k_j})_E$ is 4-isomorphic to Z.
Thus Z_J is 4-isomorphic to Z for all J.

Proposition

Let X be a space with a non-hilbertian symmetric pure basis E such that

- *$X(X)$ (i.e., the E-sum of infinitely many copies of X) is isomorphic to X, and*
- *X contains uniformly complemented and uniformly isomorphic copies of ℓ_2^n.*

Then X is isomorphic to $Z = (\sum_{k=1}^{\infty} \oplus \ell_2^k)_E$ and hence X has a continuum of renormings whose isometry groups are maximal and are not pairwise conjugate in the isomorphism group of X.

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Question

Does there exist a separable Banach space X with a unique, up to conjugacy, maximal bounded subgroup of $GL(X)$?
If yes, does X have to be isomorphic to a Hilbert space?

Thank you.