The Bishop-Phelps-Bollobás Theorem on bounded closed convex sets

Dong Hoon Cho
(joint work with Y.S. Choi)

Pohang University of Science of Technology

Besancon
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1. Bishop-Phelps Theorem

· Let C be a closed convex subset of a Banach space X. If $f(x_0) = \sup f(C)$ for some $f \in X^*$, then x_0 is called a support point.

· $T \in \mathscr{L}(X,Y)$ is called norm attaining operator if there exists $x_0 \in S_X$ such that $\|T(x_0)\| = \sup\{\|T(x)\| : x \in B_X\}$.

· (V. Klee, 1958) does every bounded closed convex subset of a Banach space have a support point?

Theorem (Bishop and Phelps, 1961)

For every Banach space X, each linear functional on X can be approximated by norm attaining ones. If X is a real Banach space and D is a bounded closed convex subset of X, then the set of linear functionals that attain their suprema on D is dense in X^*.

Theorem (Lomonosov, 2000)

There exists a bounded closed convex subset in a predual of $\mathscr{H}^\infty$ whose support functionals are zero.

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2. Radon Nykodým property and Bishop-Phelps property

Definition

Let (Ω, Σ, μ) be a finite measure space. X has *RNP* if $G : \Sigma \to X$ a μ-continuous vector measure of bounded variation, then there exists a Bochner integrable g such that $G(E) = \int_E g d\mu$ for every $E \in \Sigma$.

Definition

A subset M of X is said to be dentable if for every $\epsilon > 0$ there exists a point $x \in M$ such that $x \notin \overline{co}(M \backslash B(x, \epsilon))$.

X is dentable $\Rightarrow$ X has *RNP*. (M. Rieffel, 1967)
X has *RNP* $\Rightarrow$ X is dentable. (W. Davis, R. Phelps and R. Huff, 1974).

Theorem (Bourgain, 1978)

X has RNP if and only if for every bounded closed and symmetric convex subset D of X and for every Banach space Y, the subset of $\mathcal{L}(X, Y)$ attaining their suprema norm on D is dense in $\mathcal{L}(X, Y)$.

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3. Bishop-Phelps-Bollobás property

Theorem (Bollobás, 1970)

Let X be a (real or complex) Banach space. Given $x \in S_X$ *and* $x^* \in S_{X^*}$ *with* $|1 - x^*(x)| < \frac{\epsilon^2}{2}$*, there are elements* $y \in S_X$ *and* $y^* \in S_{X^*}$ *such that*

$$y^*(y) = 1, \quad \|x - y\| < \epsilon \quad \textit{and} \quad \|y^* - x^*\| < \epsilon + \epsilon^2.$$

Definition (Acosta et al, 2008)

Let X and Y be real or complex Banach spaces. We say that the couple (X, Y) has the Bishop-Phelps-Bollobás property for operators ($BPBP$), if given $\epsilon > 0$ there exist $\beta(\epsilon) > 0$ and $\eta(\epsilon) > 0$ with $\lim_{\epsilon \to 0^+} \beta(\epsilon) = 0$ such that for $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\epsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions :

$$\|Su_0\| = 1, \quad \|x_0 - u_0\| < \beta(\epsilon) \quad \text{and} \quad \|T - S\| < \epsilon$$

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Positive results on the pair (X,Y)

1. M.D. Acosta, R.M. Aron, D. García and M. Maestre (2008)
   If X and Y are finite dimensional, then (X,Y) has *BPBP*.
   (ℓ_1, Y) has BPBP if and only if Y has *AHSP*.
2. R.M. Aron, Y.S.Choi, D.García and M. Maestre (2011)
   $(L_1(\mu), L_\infty[0,1])$ has *BPBP*.
3. Y.S.Choi, S.K.Kim, H.J.Lee and M.Martín (2013)
   $(L_1(\mu), L_1(\nu))$ and $(L_1(\mu), L_\infty(\nu))$ have *BPBP*.
4. R.M.Aron, Cascales and Kozhushkina (2011)
   For a locally compact Hausdorff space L and Asplund space X, $(X, C_0(L))$ has *BPBP*.
5. S.K.Kim and H.J.Lee (2012)
   If X is uniformly convex space, then (X,Y) has *BPBP* for every Y.
   $(C_0(S), C_0(K))$ has *BPBP* (2013).
6. M.D.Acosta, J.Geurrero, D.García, M.Maestre (2013)
   (ℓ_1, Y) has *BPBP* for bilinear forms if and only if (Y, Y^*) has *AHSP*

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(Question) *RNP* implies *BPBP*?
(Answer) No.
(ℓ_1, Y), Y is strictly convex not uniformly convex space.

(Question) *BPBP* implies *RNP*?
(Answer) No.
$(L_1(\mu), L_\infty[0,1])$.

(Question) What is X such that (X, Y) has *BPBP* on every symmetric bounded closed convex subset $D \subset X$ for every Banach space Y?
(Answer) X is $\mathbb{R}$ if and only if X satisfies the question.

(Question) Is there any Banach space Y such that for every Banach space X, (X, Y) has *BPBP* on every symmetric bounded closed convex subset $D \subset X$?
(Answer) Y has property β.

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4. Bishop-Phelps-Bollobás property on bounded closed convex sets

Theorem (Stegall, 1978)

Let D be a bounded closed convex subset of RNP space and $f : D \to \mathbb{R}$ that is upper semi-continuous and bounded above. Then for every $\delta > 0$, there exists $x^* \in X^*$ with $\|x^*\| < \delta$ such that $f + x^*$ and $f + |x^*|$ strongly expose D.

Theorem (Cho and Choi)

Let D be a bounded closed convex set in a real Banach space X. Given $0 < \epsilon < \frac{1}{4}$ and $f \in X^*$, there exist $x^* \in X^*$ and $x_0 \in D$ such that both $f + x^*$ and $f + |x^*|$ attain their suprema on D simultaneously at x_0 and $\|x^*\| < \epsilon$. Moreover $(f + x^*)(x_0) = (f + |x^*|)(x_0)$.

Theorem (Cho and Choi)

Let D be a bounded closed convex set in a real Banach space X. Given $f \in X^*$ and $\epsilon > 0$, there exists $x^* \in X^*$ such that $|f + x^*|$ attains its supremum on D and $\|x^*\| < \epsilon$. Moreover, if D is symmetric, and $f(x_0) > \|f\|_D - \frac{\delta}{2}$ for some $x_0 \in D$ and $\delta > 0$, then $x_1 \in D$ can be chosen so that $\|x_0 - x_1\| < \frac{\delta}{\epsilon}$ and $|f + x^*|$ attains its supremum at x_1.

(Question) Is symmetry of D necessary for the location of point?
(Answer) Yes.
Choose $f \in S_{X^*}$ which does not attain its norm and let

$$S_1 = \{x \in B_X : f(x) \geq 1 - \epsilon^2\}, \quad T = B_X \cap \ker f \;\text{ and }\; S_2 = \overline{(-S_1 + T)}.$$

If we set $D = \overline{co}(S_1 \cup S_2)$ and we choose $x_0 \in D \cap S_X$ such that $f(x_0) > 1 - \frac{\epsilon^2}{2}$, then supremum norm attaining point z should be far away from x_0 i.e. $\|z - x_0\| > 2 - \epsilon$.

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Definition

A Banach space Y is called to have property β if there is $0 \leq \lambda < 1$ and a family $\{(y_\alpha, f_\alpha) \in B_Y \times B_{Y^*}\}$ such that (i) $f_\alpha(x_\alpha) = \|x_\alpha\| = 1$ (ii) $|f_\alpha(y_\beta)| \leq \lambda$ for $\alpha \neq \beta$ (iii) $\|y\| = \sup_\alpha |f_\alpha(y)|$.

· (Lindenstrauss, 1963) Y has property $\beta \Rightarrow (X, Y)$ has *BPP* on B_X.

· (Acosta et al, 2008) Y has property $\beta \Rightarrow (X, Y)$ has *BPBP* on B_X.

Theorem (Cho and Choi)

Let Y be a Banach space with property β and D be a bounded closed convex set in X. Then (X, Y) has BPP on D for every Banach space X. Moreover if D is symmetric, then (X, Y) has BPBP on D.

(Question) property β (for range space) is the counterpart of *RNP* (for domain space) with respect to *BPP*?

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$$0 < \epsilon < \frac{1-\lambda}{2+\lambda} \text{ and } \frac{\epsilon(2+\lambda)}{1-2\epsilon-\lambda-\lambda\epsilon} \leq \eta \leq \frac{2\epsilon(2+\lambda)}{1-2\epsilon-\lambda-\lambda\epsilon}.$$

Assume that $T \in \mathcal{L}(X,Y)$, $\|T\|_D = 1$, $\|T\| = M$ and $\|T(x_0)\| > 1 - \frac{\epsilon^2}{2}$ for some $x_0 \in D$. We can choose α_0 so that $|(T^* f_{\alpha_0})(x_0)| = |f_{\alpha_0}(Tx_0)| > 1 - \frac{\epsilon^2}{2}$. By the previous theorem, there exist $g \in X^*$ and $z_0 \in D$ such that

$$|g(z_0)| = \|g\|_D, \quad \|g - T^* f_{\alpha_0}\| \leq \epsilon \text{ and } \|x_0 - z_0\| \leq \epsilon.$$

Then $1 - 2\epsilon \leq 1 - \frac{\epsilon^2}{2} - \epsilon \leq \|g\|_D \leq 1 + \epsilon$. Define $T_0 \in \mathcal{L}(X,Y)$ by

$$T_0(x) = T(x) + ((1+\eta)g(x) - T^* f_{\alpha_0}(x))y_{\alpha_0}.$$

Clearly, $\|T_0^* f_{\alpha_0}\|_D = (1+\eta)\|g\|_D \geq (1+\eta)(1-2\epsilon)$.

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Clearly, $\|T_0^* f_{\alpha_0}\|_D = (1+\eta)\|g\|_D \geq (1+\eta)(1-2\epsilon)$.

proof Wlog consider $D \subset B_X$ and let

$$0 < \epsilon < \frac{1-\lambda}{2+\lambda} \text{ and } \frac{\epsilon(2+\lambda)}{1-2\epsilon-\lambda-\lambda\epsilon} \leq \eta \leq \frac{2\epsilon(2+\lambda)}{1-2\epsilon-\lambda-\lambda\epsilon}.$$

Assume that $T \in \mathscr{L}(X,Y)$, $\|T\|_D = 1$, $\|T\| = M$ and $\|T(x_0)\| > 1 - \frac{\epsilon^2}{2}$ for some $x_0 \in D$. We can choose α_0 so that $|(T^* f_{\alpha_0})(x_0)| = |f_{\alpha_0}(Tx_0)| > 1 - \frac{\epsilon^2}{2}$. By the previous theorem, there exist $g \in X^*$ and $z_0 \in D$ such that

$$|g(z_0)| = \|g\|_D, \quad \|g - T^* f_{\alpha_0}\| \leq \epsilon \text{ and } \|x_0 - z_0\| \leq \epsilon.$$

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For $\alpha \neq \alpha_0$,

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Therefore T_0 attains its supremum at $z_0 \in D$ with $\|x_0 - z_0\| \leq \epsilon$ and

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Definition

A Banach space X is Asplund space if X^* has *RNP*. Equivalently, every w^*-compact subset of (X^*, w^*) is $\|\cdot\|$-fragmentable. i.e. for every nonempty bounded subset of $A \subset X^*$ and for every $\epsilon > 0$, there is nonempty w^*-open neighborhood $V \subset X^*$ such that $A \cap V \neq \emptyset$ has $\|\cdot\|$-diameter less than ϵ.

Theorem (Aron et al, 2011)

Let $T : X \to C_0(L)$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \epsilon < \frac{1}{2}$ and $x_0 \in S_X$ such that $\|T(x_0)\| > 1 - \frac{\epsilon^2}{4}$. Then there are $u_0 \in S_X$ and an Asplund operator $S \in S_{\mathcal{L}(X, C_0(L))}$ satisfying

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Theorem (Cho and Choi)

Let X be an Asplund space and D be a bounded closed convex subset of X. If D is symmetric, then $(X, C_0(L))$ has BPBP on D, otherwise, $(X, C_0(L))$ has BPP.

(Question) Characterization Y such that (X, Y) has *BP(B)P* on D for Asplund space X.

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Theorem (Kim and Lee, 2012)

Let X be a uniformly convex Banach space and Y be a Banach space. Then (X,Y) has BPBP on B_X.

(Question) Is uniformly convex space a universal *BPB* domain on every bounded closed convex sets?
(Answer) No. Let $Y_k = \mathbb{R}^2$ with the norm

$$\|(x,y)\| = \max\left\{|x|, |y| + \frac{1}{k}|x|\right\}.$$

If $Y = [\bigotimes_{k=1}^{\infty} Y_k]_{\ell_\infty}$, then (ℓ_2^2, Y) fails to have *BPBP* on $B_{\ell_1^2}$.

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Definition

Let D be a bounded closed convex absorbing convex set in a Banach space X. For each $\epsilon > 0$, we call $\delta_D(\epsilon)$ by a modulus of convexity of D if

$$\delta_D(\epsilon) = \inf\left\{\frac{1}{2}\rho_D(x) + \frac{1}{2}\rho_D(y) - \rho_D\left(\frac{x+y}{2}\right) : x,y \in D, \quad \rho_D(x-y) \geq \epsilon\right\}.$$

Theorem (Cho and Choi)

Let X and Y be a (real or complex) Banach space and D be a bounded closed absorbing convex subset of B_X such that $\delta_D(\epsilon) > 0$ for every $0 < \epsilon < \frac{1}{2}$. If $T \in S_{\mathcal{L}(X,Y)}$ and $x_1 \in D$ satisfy

$$\|Tx_1\| > \|T\|_D - \epsilon^3 \delta_D(\epsilon)$$

for sufficiently small ϵ relatively to $\|T\|_D$, then there exist $S \in \mathcal{L}(X,Y)$ and $z \in D$ such that $\|Sz\| = \|S\|_D$, $\|S - T\| < \frac{4\epsilon^2}{1-\epsilon}$ and $\|x_1 - z\| \leq \rho_D(x_1 - z) < \frac{\epsilon}{1-\epsilon}$.

Thank you!