Supports in Lipschitz-free spaces and applications to extremal structure

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(joint work with Eva Pernecká,
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Lipschitz spaces

Let (M, d) be a complete metric space.
Fix a base point $0 \in M$.
The *Lipschitz constant* of $f : M \to \mathbb{R}$ is

$$\|f\|_L := \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x \neq y \in M \right\} .$$

The spaces of Lipschitz functions on M are

$$\mathrm{Lip}(M) = \{f : M \to \mathbb{R} : \|f\|_L < \infty\}$$
$$\mathrm{Lip}_0(M) = \{f : M \to \mathbb{R} : \|f\|_L < \infty, f(0) = 0\}$$

$\mathrm{Lip}_0(M)$ is a Banach space with norm $\|\cdot\|_L$.

Lipschitz-free spaces

For $x \in M$, consider the evaluation operators

$$\delta(x) : f \mapsto f(x).$$

Then $\delta : M \to \mathrm{Lip}_0(M)^*$ is a (nonlinear) isometric embedding.

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Lipschitz-free space

$$\mathcal{F}(M) = \overline{\operatorname{span}}\, \delta(M) \subset \mathrm{Lip}_0(M)^*$$

Theorem (Arens, Eells 1956)

$$\mathcal{F}(M)^* \cong \mathrm{Lip}_0(M)$$

Lipschitz-free subspaces

Theorem (MacShane, 1934)

Let $M_0 \subset M$. Then every $f : M_0 \to \mathbb{R}$ can be extended to M in such a way that $\|f\|_L$ and $\|f\|_\infty$ are preserved.

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Theorem (Kadets 1985)

If $M_0 \subset M$, then $\mathcal{F}(M_0) \subset \mathcal{F}(M)$ isometrically:

$$\mathcal{F}(M_0) \cong \overline{\operatorname{span}}\, \delta(M_0)$$

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$$\mathcal{F}(M_0) \cong \overline{\operatorname{span}}\, \delta(M_0)$$

We will assume $0 \in M_0$. Otherwise, we mean

$$\mathcal{F}(M_0) \equiv \mathcal{F}(M_0 \cup \{0\}).$$

The intersection theorem

Theorem (Aliaga, Pernecká 2019)

Let $K_i \subset M$ be closed subsets. Then

$$\bigcap_i \mathcal{F}(K_i) = \mathcal{F}\left(\bigcap_i K_i\right) .$$

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Let $m \in \mathcal{F}(M)$. We define the *support of m* as

$$\mathrm{supp}(m) = \bigcap \{S \subset M \text{ closed} : m \in \mathcal{F}(S)\} .$$

By the Intersection Theorem, $m \in \mathcal{F}(\mathrm{supp}(m))$.

Supports in $\mathcal{F}(M)$

Proposition

Let $m \in \mathcal{F}(M)$, $K \subset M$ closed. TFAE:

- $\operatorname{supp}(m) \subset K$
- $m \in \mathcal{F}(K)$
- If $f, g \in \mathrm{Lip}_0(M)$ satisfy $f|_K = g|_K$, then $\langle m, f\rangle = \langle m, g\rangle$

Proposition

Let $m \in \mathcal{F}(M)$, $p \in M$. TFAE:

- $p \in \operatorname{supp}(m)$
- For every neighborhood U of p, there is $f \in \mathrm{Lip}_0(M)$ supported on U such that $\langle m, f\rangle \neq 0$

Weighting in $\mathrm{Lip}_0(M)$ and $\mathcal{F}(M)$

Proposition

Let $h \in \mathrm{Lip}(M)$ with bounded support. If $f \in \mathrm{Lip}_0(M)$ then $f \cdot h \in \mathrm{Lip}_0(M)$ and

$$T_h : f \mapsto f \cdot h$$

is a w^*-w^*-continuous linear operator on $\mathrm{Lip}_0(M)$.

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is a w^*-w^*-continuous linear operator on $\mathrm{Lip}_0(M)$.

Thus T_h has a continuous preadjoint

$$(T_h)_* : \mathcal{F}(M) \to \mathcal{F}(M)$$

defined by $(T_h)_*(m) = m \circ T_h \in \mathcal{F}(M)$:

$$\langle m \circ T_h, f \rangle = \langle m, T_h(f) \rangle = \langle m, f \cdot h \rangle \quad \text{for } f \in \mathrm{Lip}_0(M)$$

Extremal structure of $\mathcal{F}(M)$

Research program

Let M be a complete pointed metric space.
What are the extreme points of $B_{\mathcal{F}(M)}$?

Using these techniques, we characterize

- extreme points of the positive part of $B_{\mathcal{F}(M)}$
- extreme points with finite support
- exposed points with finite support
- extreme points of the form (positive + finite support)

Extreme points of the positive ball

Theorem (Aliaga, Petitjean, Procházka 2019)

The extreme points of $B_{\mathcal{F}(M)} \cap \mathcal{F}(M)^+$ are $\dfrac{\delta(p)}{\|\delta(p)\|}$, $p \in M$.

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Proof of necessity: Let m be extreme. Suppose $p, q \in \operatorname{supp}(m)$.
Let $r = d(p,q)$ and $h \in \operatorname{Lip}(M)$ such that

- $h(x) = 1$ if $d(x,p) \leq r/4$
- $h(x) = 0$ if $d(x,p) \geq r/2$
- $0 \leq h(x) \leq 1$ in between

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Then $m \circ T_h$ and $m - m \circ T_h = m \circ T_{1-h}$ are in $\mathcal{F}(M)^+$ and $\neq 0$.
So $\|m \circ T_h\| + \|m \circ T_{1-h}\| = \|m\| = 1$ and

$$m = \|m \circ T_h\| \frac{m \circ T_h}{\|m \circ T_h\|} + \|m \circ T_{1-h}\| \frac{m \circ T_{1-h}}{\|m \circ T_{1-h}\|}$$

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The extreme points of $B_{\mathcal{F}(M)} \cap \mathcal{F}(M)^+$ are $\frac{\delta(p)}{\|\delta(p)\|}$, $p \in M$.

Proof of sufficiency: Suppose $\delta(p) = m_1 + m_2$ with $m_i \in \mathcal{F}(M)^+$.
Suppose $q \in \operatorname{supp}(m_1)$, $q \neq p$.

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Suppose $q \in \operatorname{supp}(m_1)$, $q \neq p$.
Let U be a ball around q such that $p \notin U$.
Then there is $f \in \operatorname{Lip}_0(M)$ with $\operatorname{supp}(f) \subset U$, $f \geq 0$, $\langle m_1, f \rangle > 0$.
But then

$$\langle m_2, f \rangle = \langle \delta(p), f \rangle - \langle m_1, f \rangle = -\langle m_1, f \rangle < 0$$

contradicts $m_2 \geq 0$. $\square$

Elementary molecules

An *elementary molecule* is $u_{pq} = \dfrac{\delta(p) - \delta(q)}{d(p,q)} \in S_{\mathcal{F}(M)}$.

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Properties:

- $\|f\|_L = \sup\left\{\langle u_{pq}, f\rangle : p, q \in M\right\}$ since

$$\langle u_{pq}, f\rangle = \frac{f(p) - f(q)}{d(p,q)}$$

- $B_{\mathcal{F}(M)} = \overline{\operatorname{conv}}\left\{u_{pq} : p, q \in M\right\}$
- For every $\varepsilon > 0$, $m \in \mathcal{F}(M)$ admits an expression

$$m = \sum_{n=1}^{\infty} a_n u_{p_n q_n} \quad \text{where} \quad \sum_{n=1}^{\infty} |a_n| < \|m\| + \varepsilon$$

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Theorem (Weaver 1995)

Every preserved extreme point of $B_{\mathcal{F}(M)}$ is an elementary molecule.

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Corollary

Every extreme point of $B_{\mathcal{F}(M)}$ with finite support is an elementary molecule.

Extreme molecules

Let $p, q \in M$. The *metric segment* between p and q is

$$[p, q] = \{x \in M : d(p, x) + d(x, q) = d(p, q)\} \, .$$

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If u_{pq} is an extreme point of $B_{\mathcal{F}(M)}$, then $[p, q] = \{p, q\}$.

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If u_{pq} is an extreme point of $B_{\mathcal{F}(M)}$, then $[p, q] = \{p, q\}$.

Proof: If $x \in [p, q]$ then $u_{pq} \in [u_{px}, u_{xq}]$:

$$u_{pq} = \frac{\delta(p) - \delta(q)}{d(p, q)} = \frac{\delta(p) - \delta(x)}{d(p, q)} + \frac{\delta(x) - \delta(q)}{d(p, q)}$$
$$= \frac{d(p, x)}{d(p, q)} u_{px} + \frac{d(x, q)}{d(p, q)} u_{xq}. \qquad \square$$

Extreme molecules

Theorem (Aliaga, Pernecká 2018)

u_{pq} is an extreme point of $B_{\mathcal{F}(M)}$ iff $[p,q] = \{p,q\}$.

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The intersection of all faces of $B_{\mathcal{F}(M)}$ that contain u_{pq} is contained in $\mathcal{F}([p,q])$.

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Proof: Let $m \in S_{\mathcal{F}(M)}$ belong to all faces containing u_{pq}. Fix $\delta, \varepsilon > 0$ and let

$$[p,q]_\varepsilon = \{x \in M : d(p,x) + d(x,q) \leq d(p,q) + \varepsilon\}$$

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$$[p,q]_\varepsilon = \{x \in M : d(p,x) + d(x,q) \leq d(p,q) + \varepsilon\}$$

Let $m = \sum_{n=1}^{\infty} a_n u_{x_n y_n}$ where $\sum_{n=1}^{\infty} |a_n| < 1 + \delta$.

Let $m' = \sum_{x_n, y_n \in [p,q]_\varepsilon} a_n u_{x_n y_n} \in \mathcal{F}([p,q]_\varepsilon)$.

Sketch of proof

Theorem (Aliaga, Pernecká 2018)

The intersection of all faces of $B_{\mathcal{F}(M)}$ that contain u_{pq} is contained in $\mathcal{F}([p,q])$.

There is a constant $C < \infty$ depending on ε such that

$$\left\|m - m'\right\| < C \cdot \delta$$

Thus m is $(C \cdot \delta)$-close to $\mathcal{F}([p,q]_\varepsilon)$.

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Thus $m \in \mathcal{F}([p,q]_\varepsilon)$.
By the Intersection Theorem,

$$m \in \bigcap_{\varepsilon>0} \mathcal{F}([p,q]_\varepsilon) = \mathcal{F}\left(\bigcap_{\varepsilon>0} [p,q]_\varepsilon\right) = \mathcal{F}([p,q]). \quad \square$$

Exposed molecules

Theorem (Petitjean, Procházka 2018)

If $[p,q] = \{p,q\}$ then u_{pq} is an exposed point of $B_{\mathcal{F}(M)}$.

$x \in S_X$ is *exposed* $\equiv$ there is $f \in S_{X^*}$ that attains its norm at x and nowhere else.

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The exposing functional for u_{pq} is

$$f_{pq}(x) = \frac{d(p,q)}{2} \frac{d(x,q) - d(x,p)}{d(x,q) + d(x,p)} - \text{constant}$$

with the property:

$$\langle u_{xy}, f_{pq} \rangle \geq 1 - \varepsilon \quad \text{implies} \quad x, y \in [p,q]_\varepsilon$$

where

$$[p,q]_\varepsilon = \{x \in M : d(p,x) + d(x,q) \leq d(p,q) + \varepsilon\}$$

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Proof: Let $m \in S_{\mathcal{F}(M)}$ with $\langle m, f_{pq} \rangle = 1$. Fix $\delta, \varepsilon > 0$ and write

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Let $I = \left\{ n : \langle u_{x_n y_n}, f_{pq} \rangle \geq 1 - \varepsilon \right\}$. Then

$$1 = \langle m, f_{pq} \rangle = \sum_{n=1}^{\infty} a_n \langle u_{x_n y_n}, f_{pq} \rangle = \sum_{n \in I} + \sum_{n \notin I}$$
$$\leq \sum_{n \in I} |a_n| + (1 - \varepsilon) \sum_{n \notin I} |a_n| < (1 + \delta) - \varepsilon \sum_{n \notin I} |a_n| \,.$$

Exposed molecules

Theorem (Petitjean, Procházka 2018)

The face of $B_{\mathcal{F}(M)}$ determined by f_{pq} is contained in $\mathcal{F}([p,q])$.

So

$$\left\| m - \sum_{n\in I} a_n u_{x_n y_n} \right\| = \left\| \sum_{n\notin I} a_n u_{x_n y_n} \right\| \leq \sum_{n\notin I} |a_n| < \frac{\delta}{\varepsilon}.$$

Thus m is (δ/ε)-close to $\sum_{n\in I} a_n u_{x_n y_n} \in \mathcal{F}([p,q]_\varepsilon)$.

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Thus $m \in \mathcal{F}([p,q]_\varepsilon)$.
By the Intersection Theorem,

$$m \in \bigcap_{\varepsilon>0} \mathcal{F}([p,q]_\varepsilon) = \mathcal{F}\left(\bigcap_{\varepsilon>0} [p,q]_\varepsilon\right) = \mathcal{F}([p,q]). \quad \square$$

Almost positive extreme points

Theorem (Aliaga, Petitjean, Procházka 2019)

If an extreme point m of $B_{\mathcal{F}(M)}$ has the form as $m = \pi + \phi$ where $\pi \in \mathcal{F}(M)^+$ and ϕ has finite support, then m is a molecule.

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Lemma

Fix $\phi \in \mathcal{F}(M)$, $S = \mathrm{supp}(\phi) \cup \{0\}$. Let $\pi \in \mathcal{F}(M)^+$ and define

$$N_\pi(f) = \langle \pi + \phi, E(f) \rangle$$

for $f \in B_{\mathrm{Lip}_0(S)}$, where $E(f) \in B_{\mathrm{Lip}_0(M)}$ is

$$E(f)(x) = \inf_{p \in S} \{f(p) + d(x,p)\}.$$

Then N_π is concave and its maximum is $\|\pi + \phi\|$.

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Then N_π is concave and its maximum is $\|\pi + \phi\|$.

$E(f)$ is the largest extension of $f \in B_{\mathrm{Lip}_0(S)}$ to $B_{\mathrm{Lip}_0(M)}$.

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Fix $f_0 \in B_{\mathrm{Lip}_0(S)}$ such that $N_\pi(f_0) = \|\pi + \phi\|$.
If $\mathrm{supp}(\pi)$ is not finite, we will construct $\nu \in \mathcal{F}(M)$ such that

(1) $\pi \pm \nu$ are positive

(2) $\langle \nu, E(f) \rangle = 0$ for all $f \in \mathrm{Lip}_0(S)$ in a neighborhood of f_0

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Then

$$N_{\pi\pm\nu}(f) = N_\pi(f) \pm \langle \nu, E(f) \rangle = N_\pi(f)$$

has a local maximum at f_0, hence a global maximum, hence

$$\|(\pi + \phi) \pm \nu\| = N_{\pi\pm\nu}(f_0) = N_\pi(f_0) = \|\pi + \phi\| .$$

So $\pi + \phi$ is not extreme.

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Proof of the theorem for $\phi = \delta(p)$:

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Let

$$\begin{aligned} A^+ &= \{x : d(x,0) < f_0(p) + d(x,p)\} \\ A^0 &= \{x : d(x,0) = f_0(p) + d(x,p)\} \\ A^- &= \{x : d(x,0) > f_0(p) + d(x,p)\} \end{aligned}$$

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Suppose there are $q_1, q_2, q_3 \in \operatorname{supp}(\pi) \cap A^+$.
Find disjoint balls B_i with $q_i \in B_i \subset A^+$.
Find functions $h_i \geq 0$ with $\operatorname{supp}(h_i) \subset B_i$ and $\langle \pi, h_i \rangle > 0$.
Find constants $c_i \neq 0$ such that $\|c_i h_i\|_\infty < 1$ and

$$\begin{aligned} c_1 \langle \pi, h_1 \rangle + c_2 \langle \pi, h_2 \rangle + c_3 \langle \pi, h_3 \rangle &= 0 \\ c_1 \langle \pi, h_1 \cdot E(f_0) \rangle + c_2 \langle \pi, h_2 \cdot E(f_0) \rangle + c_3 \langle \pi, h_3 \cdot E(f_0) \rangle &= 0 \end{aligned}$$

Take $\nu = \pi \circ T_h$ where $h = c_1 h_1 + c_2 h_2 + c_3 h_3$.

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$$c_1 \langle \pi, h_1 \cdot E(f_0) \rangle + c_2 \langle \pi, h_2 \cdot E(f_0) \rangle + c_3 \langle \pi, h_3 \cdot E(f_0) \rangle = 0$$

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Suppose there are $q_1, q_2, q_3 \in \operatorname{supp}(\pi) \cap A^+$ (or A^-).
Find disjoint balls B_i with $q_i \in B_i \subset A^+$ (or A^-).
Find functions $h_i \geq 0$ with $\operatorname{supp}(h_i) \subset B_i$ and $\langle \pi, h_i \rangle > 0$.
Find constants $c_i \neq 0$ such that $\|c_i h_i\|_\infty < 1$ and

$$c_1 \langle \pi, h_1 \rangle + c_2 \langle \pi, h_2 \rangle + c_3 \langle \pi, h_3 \rangle = 0$$

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Almost positive extreme points

Proof of the theorem for $\phi = \delta(p)$:

$$E(f)(x) = \min\{f(0) + d(x,0), f(p) + d(x,p)\}$$

Otherwise let $q_1, q_2, q_3 \in \mathrm{supp}(\pi) \cap A^0$.
Find disjoint balls B_i with $q_i \in B_i$ and $B_i \cap \mathrm{supp}(\pi) \setminus A^0 = \varnothing$.
Find functions $h_i \geq 0$ with $\mathrm{supp}(h_i) \subset B_i$ and $\langle \pi, h_i \rangle > 0$.
Find constants $c_i \neq 0$ such that $\|c_i h_i\|_\infty < 1$ and

$$\begin{aligned} c_1 \langle \pi, h_1 \rangle + c_2 \langle \pi, h_2 \rangle + c_3 \langle \pi, h_3 \rangle &= 0 \\ c_1 \langle \pi, h_1 \cdot E(f_0) \rangle + c_2 \langle \pi, h_2 \cdot E(f_0) \rangle + c_3 \langle \pi, h_3 \cdot E(f_0) \rangle &= 0 \end{aligned}$$

Take $\nu = \pi \circ T_h$ where $h = c_1 h_1 + c_2 h_2 + c_3 h_3$. $\square$

Open problem

Main question

Let M be a complete pointed metric space.
Are all extreme points of $B_{\mathcal{F}(M)}$ molecules?

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- M is compact and $\mathcal{F}(M) = \mathrm{lip}_0(M)^*$ (Weaver 1999)
    - M is compact Hölder
    - M is compact and countable (Dalet 2015)
    - M is compact and ultrametric (Dalet 2015)

$$\mathrm{lip}_0(M) = \left\{ f \in \mathrm{Lip}_0(M) : \frac{|f(p) - f(q)|}{d(p,q)} \to 0 \text{ unif. as } d(p,q) \to 0 \right\}$$

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- M is a subset of an $\mathbb{R}$-tree
  (Petitjean, Procházka 2019)

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Equivalently,

(1) Are all extreme points of $B_{\mathcal{F}(M)}$ exposed?
(2) Are all exposed points of $B_{\mathcal{F}(M)}$ molecules?

Thank you for your attention!

References:

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