Locally $C^{1,1}$ convex extensions of jets and Lusin properties of convex functions

Daniel Azagra

UNIVERSIDAD COMPLUTENSE MADRID

ICMAT INSTITUTO DE CIENCIAS MATEMÁTICAS

Banach spaces and optimization:
Conference on the occasion of Robert Deville's 60th birthday
Metabief, June 2019

Two related problems

Let $\mathcal{C}$ be a class of differentiable functions on $\mathbb{R}^n$. For instance, $\mathcal{C} = C^1(\mathbb{R}^n)$, $C^2(\mathbb{R}^n)$, $C^{1,1}(\mathbb{R}^n)$, or $C^{1,1}_{\text{loc}}(\mathbb{R}^n)$.

Problem (Analytical version)

Given E a subset of $\mathbb{R}^n$, and two functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, how can we tell whether there is a *convex* function $F : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}$ such that $(F, \nabla F) = (f, G)$ on E?

Let $\mathcal{C}$ be a class of differentiable functions on $\mathbb{R}^n$. For instance, $\mathcal{C} = C^1(\mathbb{R}^n)$, $C^2(\mathbb{R}^n)$, $C^{1,1}(\mathbb{R}^n)$, or $C^{1,1}_{\text{loc}}(\mathbb{R}^n)$.

Problem (Analytical version)

Given E a subset of $\mathbb{R}^n$, and two functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, how can we tell whether there is a *convex* function $F : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}$ such that $(F, \nabla F) = (f, G)$ on E?

Problem (Geometrical version)

Given an arbitrary subset C of $\mathbb{R}^n$ and a collection $\mathcal{H}$ of affine hyperplanes of $\mathbb{R}^n$ such that every $H \in \mathcal{H}$ passes through some point $x_H \in C$, what conditions on $\mathcal{H}$ are necessary and sufficient for the existence of a *convex* hypersurface of class $\mathcal{C}$ such that H is tangent to S at x_H for every $H \in \mathcal{H}$?

Previous results on smooth convex extension

M. Ghomi (2002) and M. Yan (2013) considered the problem: if $f : C \subset \mathbb{R}^n \to \mathbb{R}$ is convex and of class C^m on a neighborhood of C (so in fact we are given an m-jet on C), when does there exist a *convex* function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C?

One of the main obstacles when trying to apply Whitney extension techniques in the convex case is the fact that partitions of unity destroy convexity.
This difficulty can be overcome, as these authors showed, if C is convex and compact and $D^2 f > 0$ on ∂C.

These are very strong assumptions.

M. Ghomi (2002) and M. Yan (2013) considered the problem: if $f : C \subset \mathbb{R}^n \to \mathbb{R}$ is convex and of class C^m on a neighborhood of C (so in fact we are given an m-jet on C), when does there exist a *convex* function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C?

One of the main obstacles when trying to apply Whitney extension techniques in the convex case is the fact that partitions of unity destroy convexity.
This difficulty can be overcome, as these authors showed, if C is convex and compact and $D^2 f > 0$ on ∂C.

These are very strong assumptions.

More recent results: for the case $\mathcal{C} = C^\infty(\mathbb{R}^n)$, or $\mathcal{C} = C^m(\mathbb{R}^n)$, $m \geq 2$, see next talk!

M. Ghomi (2002) and M. Yan (2013) considered the problem: if $f : C \subset \mathbb{R}^n \to \mathbb{R}$ is convex and of class C^m on a neighborhood of C (so in fact we are given an m-jet on C), when does there exist a *convex* function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C?

One of the main obstacles when trying to apply Whitney extension techniques in the convex case is the fact that partitions of unity destroy convexity.

This difficulty can be overcome, as these authors showed, if C is convex and compact and $D^2 f > 0$ on ∂C.

These are very strong assumptions.

More recent results: for the case $\mathcal{C} = C^\infty(\mathbb{R}^n)$, or $\mathcal{C} = C^m(\mathbb{R}^n)$, $m \geq 2$, see next talk!

For $\mathcal{C} = C^{1,1}$ we have:

Theorem (Azagra-Mudarra, 2016)

Let (f, G) be a 1-jet defined on an arbitrary subset E of $\mathbb{R}^n$. There exists $F \in C^{1,1}_{conv}$ such that $(F, \nabla F)$ extends (f, G) if and only if

$$f(x) \geq f(y) + \langle G(y), x-y\rangle + \frac{1}{2M}|G(x)-G(y)|^2 \quad \text{for all } x, y \in E, \; (CW^{1,1})$$

where

$$M = M(G,E) := \sup_{x,y\in E,\, x\neq y} \frac{|G(x)-G(y)|}{|x-y|}.$$

Theorem (Azagra-Mudarra, 2016)

Let (f, G) be a 1-jet defined on an arbitrary subset E of $\mathbb{R}^n$. There exists $F \in C^{1,1}_{conv}$ such that $(F, \nabla F)$ extends (f, G) if and only if

$$f(x) \geq f(y) + \langle G(y), x-y\rangle + \frac{1}{2M}|G(x)-G(y)|^2 \quad \text{for all } x, y \in E, \; (CW^{1,1})$$

where

$$M = M(G, E) := \sup_{x,y \in E,\, x \neq y} \frac{|G(x) - G(y)|}{|x-y|}.$$

Theorem (Azagra-Le Guyer-Mudarra, 2017)

The function

$$F(x) = conv\left(\inf_{y \in E}\{f(y) + \langle G(y), x-y\rangle + \tfrac{M}{2}|x-y|^2\}\right)$$

defines such an extension, with the property that $Lip(\nabla F) \leq M$. Moreover, the result holds for any Hilbert space in place of $\mathbb{R}^n$.

Recall that

$$\mathrm{conv}(g)(x) = \sup\{h(x) \,:\, h \text{ is convex and continuous, } h \leq g\}.$$

Other useful expressions for $\mathrm{conv}(g)$ are given by

$$\mathrm{conv}(g)(x) = \inf\left\{\sum_{j=1}^{n+1} \lambda_j g(x_j) \,:\, \lambda_j \geq 0, \sum_{j=1}^{n+1} \lambda_j = 1,\, x = \sum_{j=1}^{n+1} \lambda_j x_j\right\} \tag{2.1}$$

and by the Fenchel biconjugate of g, that is,

$$\mathrm{conv}(g) = g^{**}, \tag{2.2}$$

where

$$h^*(x) := \sup_{v \in \mathbb{R}^n} \{\langle v, x\rangle - h(v)\}. \tag{2.3}$$

Theorem (Daniilidis-Haddou-Le Gruyer-Ley, 2017)

If (f,G) is a 1-jet defined on a subset E of a Hilbert space and (f,G) satisfies $(CW^{1,1})$ for some M then the formula

$$F(x) = \sup_{\varepsilon\in(0,\frac{1}{M})} \inf_{z\in X} \sup_{y\in X} \sup_{u\in E}\{f(u) + \langle G(u), z-u\rangle - \frac{\|y-z\|^2}{2\varepsilon} + \frac{\|z-x\|^2}{2\varepsilon}\}$$

defines a $C^{1,1}(X)$ convex function such that $(F,\nabla F)$ extends (f,G), and $\mathrm{Lip}\,\nabla F \leq M$.

Theorem (Azagra-Mudarra, 2017: Finding $C^{1,1}$ convex hypersurfaces with prescribed tangent hyperplanes at a given subset of a Hilbert space)

Let E be an arbitrary subset of a Hilbert space X, and let $N : E \to S_X$ be a mapping. Then the following statements are equivalent.

1. *There exists a $C^{1,1}$ convex hypersurface S such that $E \subseteq S$ and $N(x)$ is outwardly normal to S at x for every $x \in E$.*
2. *There exists some $\delta > 0$ such that*

$$\langle N(y), y - x \rangle \geq \delta \|N(y) - N(x)\|^2 \quad \textit{for all} \quad x, y \in E.$$

Theorem (Reformulation of Azagra-LeGruyer-Mudarra's 2017 theorem)

Let E be an arbitrary nonempty subset of $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ be given functions. Assume that

$$f(z) + \langle G(z), x - z \rangle \leq f(y) + \langle G(y), x - y \rangle + \frac{M}{2}|x - y|^2 \tag{2.4}$$

for every $y, z \in E$ and every $x \in \mathbb{R}^n$. Then the formula

$$F = conv\left(x \mapsto \inf_{y \in E}\left\{f(y) + \langle G(y), x - y \rangle + \frac{M}{2}|x - y|^2\right\}\right) \tag{2.5}$$

defines a $C^{1,1}$ convex extension of f to $\mathbb{R}^n$ which satisfies $\nabla F = G$ on E and $Lip(\nabla F) \leq M$.

Geometrically speaking, the epigraph of F is the closed convex envelope in $\mathbb{R}^{n+1}$ of the union of the family of paraboloids $\{\mathcal{P}_y : y \in E\}$, where $\mathcal{P}_y = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t = f(y) + \langle G(y), x - y \rangle + \frac{M}{2}|x - y|^2, x \in \mathbb{R}^n\}$, which must lie above the putative tangent hyperplanes.

New results: the case $\mathcal{C} = C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Now we will be looking for analogues of this result for the more complicated case of $C^{1,1}_{\text{loc}}$ convex extensions of 1-jets.

If the given jet (f, G) has the property that

$$\text{span}\{G(y) - G(z) : y, z \in E\} = \mathbb{R}^n$$

(which is rather generic), then our result is easier to understand and use.

Theorem (2019)

Assume that span $\{G(x) - G(y) : x, y \in E\} = \mathbb{R}^n$. *Then there exists a convex function* $F \in C^{1,1}_{loc}(\mathbb{R}^n)$ *such that* $(F, \nabla F) = (f, G)$ *on* E *if and only if for each* $y \in E$ *there exists a (not necessarily convex)* $C^{1,1}_{loc}$ *function* $\varphi_y : \mathbb{R}^n \to [0, \infty)$ *such that:*

$$\varphi_y(y) = 0, \nabla\varphi_y(y) = 0;$$

$$\sup\left\{ \frac{|\nabla\varphi_y(x) - \nabla\varphi_y(z)|}{|x - z|} : x, z \in B(0, R), x \neq z, \ y \in E \cap B(0, R) \right\} < \infty$$

for every $R > 0$, *and*

$$f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \varphi_y(x) \quad \forall y, z \in E \ \ \forall x \in \mathbb{R}^n.$$

Whenever these conditions are satisfied, we can take (for any $a > 0$*)*

$$F = conv\left(x \mapsto \inf_{y \in E} \left\{ f(y) + \langle G(y), x - y\rangle + \varphi_y(x) + a|x - y|^2 \right\} \right).$$

The above result is quite general and may be very useful in some situations, as it gives us a lot of freedom in choosing a suitable family of functions $\{\varphi_y\}_{y\in E}$, but of course they do not tell us how to find such a family, which may be inconvenient in other situations. In order to decide whether or not such functions exist and, if they do, how to build them, we need to know something about the global behavior of at least one convex extension ψ of f satisfying $\psi(x) \geq f(y) + \langle G(y), x-y\rangle$ for all $x \in \mathbb{R}^n$ and $y \in E$. The most natural (and minimal) of such extensions is given by

$$m(x) := \sup_{y\in E}\{f(y) + \langle G(y), x-y\rangle\}.$$

The following Corollary gives us a practical condition for the existence of convex extensions F of the jet (f, G).

Corollary (2019)

Let E be an arbitrary nonempty subset of $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ be functions such that

$$span\{G(x) - G(y) : x, y \in E\} = \mathbb{R}^n.$$

Then there exists a convex function $F \in C^{1,1}_{loc}(\mathbb{R}^n)$ such that $F_{|_E} = f$ and $(\nabla F)_{|_E} = G$ if and only if for each $k \in \mathbb{N}$ there exists a number $A_k \geq 2$ such that

$$m(x) \leq f(y) + \langle G(y), x - y\rangle + \frac{A_k}{2}|x-y|^2 \quad \forall y \in E \cap B(0,k) \;\; \forall x \in B(0,4k).$$

Equivalently,

$$f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \frac{A_k}{2}|x-y|^2$$

for every $z \in E$, every $y \in E \cap B(0,k)$, and every $x \in B(0,4k)$.

If the given function G is bounded then we can obtain a much more explicit formula for the extension.

Corollary (2019)

Assume that G is bounded and span$\{G(x) - G(y) : x, y \in E\} = \mathbb{R}^n$. Then there exists a convex function $F \in C^{1,1}_{loc}(\mathbb{R}^n)$ such that $F_{|_E} = f$ and $(\nabla F)_{|_E} = G$ if and only if for each $k \in \mathbb{N}$ there exists a number A_k such that

$$A_k \geq 1 + 4 \sup_{y \in E} |G(y)|, \quad \textit{and}$$

$$m(x) \leq f(y) + \langle G(y), x - y \rangle + \frac{A_k}{2}|x-y|^2 \quad \forall y \in E \cap B(0,k) \ \ \forall x \in B(0,2k).$$

Moreover, a formula for such an extension F is given by

$$F(x) = conv\left(x \mapsto \inf_{y \in E}\{f(y) + \langle G(y), x-y \rangle + A_{k(y)}|x-y|^2\}\right),$$

where $k(y)$ is defined as the first positive integer such that $y \in B(0,k)$.

Sketch of the proof of the Theorem.

Consider

$$g(x) := \inf_{y \in E} \{f(y) + \langle G(y), x - y\rangle + \varphi_y(x)\}$$

and

$$m(x) := \sup_{z \in E} \{f(z) + \langle G(z), x - z\rangle\}.$$

Observe that m and g are finite everywhere; indeed, taking two points $y_0, z_0 \in E$, we have

$$f(z_0) + \langle G(z_0), x - z_0\rangle \leq m(x) \leq g(x) \leq f(y_0) + \langle G(y_0), x - y_0\rangle + \varphi_{y_0}(x)$$

for every $x \in \mathbb{R}^n$. In particular we have

$$m(x) \leq g(x) \text{ for all } x \in \mathbb{R}^n.$$

Besides m is obviously convex on $\mathbb{R}^n$, and by using that $\varphi_y(y) = 0, \nabla\varphi_y(y) = 0$ and the previous inequality one checks that

$$f(x) = m(x)$$

for every $x \in E$, and that f and G are bounded on bounded sets.

The condition $\mathrm{span}\{G(y) - G(z) : y, z \in E\} = \mathbb{R}^n$ implies that m is essentially coercive, that is, there exist a convex function $c : \mathbb{R}^n \to \mathbb{R}$ and a vector $v \in \mathbb{R}^n$ such that

$$m(x) = c(x) + \langle v, x \rangle \text{ for all } x \in \mathbb{R}^n,$$

with $\lim_{|x|\to\infty} c(x) = \infty$. In particular the function c attains a global minimum at some point $x_0 \in \mathbb{R}^n$. Hence, up to replacing the jet (f, G) with the jet $(\widetilde{f}, \widetilde{G})$ defined by $\widetilde{f}(y) = f(y) - c(x_0) - \langle v, y \rangle$, $\widetilde{G}(y) = G(y) - v$, and the function $m(x)$ with $c(x) - c(x_0)$, we may assume that

$$\lim_{|x|\to\infty} m(x) = \infty, \quad \text{and } m(x) \geq 0 \text{ for all } x \in \mathbb{R}^n \tag{3.1}$$

(note that any function that does not depend on y can be taken in and out of a sum in the infimum defining g, and the same goes for any affine function and the convex envelope).

From the definitions of g and m, and bearing in mind that $\varphi_y(y) = 0$ for each $y \in E$, we also obtain

$$f(x) \leq m(x) \leq g(x) \leq f(x) \text{ for every } x \in E,$$

hence

$$g(x) = m(x) = f(x) \text{ for all } x \in E.$$

Lemma

The function g is locally Lipschitz, and for every $R > 0$ there exists $C_R > 0$ such that for every $x, h \in B(0, R)$ we have

$$g(x+h) + g(x-h) - 2g(x) \leq C_R |h|^2.$$

(We omit the proof)

Next, by adapting Kirchheim-Kristensen's proof of the C^1 smoothness of the convex envelope, we may see that this kind of inequality is preserved (up to some constants) when we pass to the convex envelope.

Lemma

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $\lim_{|x|\to\infty} g(x) = \infty$ and such that for every $R > 0$ there exists $C_R > 0$ so that for every $x, h \in B(0, R)$ we have

$$g(x+h) + g(x-h) - 2g(x) \leq C_R |h|^2.$$

Then the function $F = conv(g)$ has a similar property: for every $R > 0$ there exists $C'_R > 0$ such that for every $x, h \in B(0, R)$ we have

$$F(x+h) + F(x-h) - 2F(x) \leq C'_R |h|^2.$$

Therefore $F \in C^{1,1}_{loc}(\mathbb{R}^n)$.

End of the proof of the Theorem:
Since m is convex, by definition of convex envelope we have

$$m \leq F \leq g \text{ on } \mathbb{R}^n,$$

which together with the already known fact that $g = m = f$ on E, allows us to conclude that $F = f$ on E.

Finally, we have $m \leq F$ on $\mathbb{R}^n$ and $F = m$ on E, where m is convex and F is differentiable on $\mathbb{R}^n$. This implies that m is differentiable on E, with $\nabla m(x) = \nabla F(x)$ for all $x \in E$. Since we obviously have $G(x) \in \partial m(x)$ for all $x \in E$, we also obtain that $\nabla F(x) = G(x)$ for all $x \in E$. □

Lusin properties of convex functions

Let $\mathcal{A}$ and $\mathcal{C}$ be two classes of functions contained in the class of all functions from $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$) into $\mathbb{R}$. If for a given $f \in \mathcal{A}$ and every $\varepsilon > 0$ we can find a function $g \in \mathcal{C}$ such that

$$\mathcal{L}^n\left(\{x : f(x) \neq g(x)\}\right) < \varepsilon, \tag{4.1}$$

we will say that f *has the Lusin property of class* $\mathcal{C}$. Here $\mathcal{L}^n$ denotes Lebesgue's outer measure in $\mathbb{R}^n$. If every function $f \in \mathcal{A}$ satisfies this property, we will also say that the $\mathcal{A}$ has the Lusin property of class $\mathcal{C}$.

This terminology comes from the well known Lusin's theorem (1912): for every Lebesgue-measurable function $f : \mathbb{R}^n \to \mathbb{R}$ and every $\varepsilon > 0$ there exists a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{L}^n\left(\{x : f(x) \neq g(x)\}\right) < \varepsilon$. That is, measurable functions have the Lusin property of class $C(\mathbb{R}^n)$.

Several authors have shown that one can take g of class C^k if f has some weaker regularity properties of order k:

- H. Federer (1944): almost everywhere differentiable functions (and in particular locally Lipschitz functions) have the Lusin property of class C^1.
- H. Whitney (1951) improved this result by showing that a function $f : \mathbb{R}^n \to \mathbb{R}$ has approximate partial derivatives of first order a.e. if and only if f has the Lusin property of class C^1.
- Calderon and Zygmund (1961) proved analogous results for $\mathcal{A} = W^{k,p}(\mathbb{R}^n)$ (the class of Sobolev functions) and $\mathcal{C} = C^k(\mathbb{R}^n)$.
- Other authors, including Liu, Bagby, Michael-Ziemer, Bojarski-Hajłasz-Strzelecki, and Bourgain-Korobkov-Kristensen have improved Calderon and Zygmund's result in several directions, by obtaining additional estimates for $f - g$ in the Sobolev norms, as well as the Bessel capacities or the Hausdorff contents of the exceptional sets where $f \neq g$.
- Generalizing Whitney's result, and Liu and Tai independently established that a function $f : \mathbb{R}^n \to \mathbb{R}$ has the Lusin property of class C^k if and only if f is approximately differentiable of order k almost everywhere.

The Whitney extension technique, or other techniques also related to Whitney cubes and associated partitions of unity, play a key role in the proofs of all of these results.

The Whitney extension technique, or other techniques also related to Whitney cubes and associated partitions of unity, play a key role in the proofs of all of these results.

For the special class of *convex* functions $f : \mathbb{R}^n \to \mathbb{R}$:
G. Alberti (1994) and S.A. Imonkulov (1992) independently showed that every convex function has the Lusin property of class C^2. However, given a convex function f and $\varepsilon > 0$, the function $g \in C^2(\mathbb{R}^n)$ satisfying $\mathcal{L}^n(\{x : f(x) \neq g(x)\}) < \varepsilon$ that they obtained is not convex. This fact is rather disappointing and may thwart the applicability of their result.

Our $C^{1,1}_{\mathrm{loc}}$ convex extension results allow us to show:

Theorem (Azagra-Hajłasz 2019)

Let $f:\mathbb{R}^n\to\mathbb{R}$ be a convex function. Then f has the Lusin property of class $C^{1,1\,loc}_{conv}(\mathbb{R}^n)$ (meaning that for every $\varepsilon>0$ there exists $g:\mathbb{R}^n\to\mathbb{R}$ convex and of class $C^{1,1}_{loc}$ with $\mathcal{L}^n\left(\{x\in\mathbb{R}^n : f(x)\neq g(x)\}\right)<\varepsilon$) if and only if:

- *either f is essentially coercive (in the sense that $\lim_{|x|\to\infty} f(x)-\ell(x)=\infty$ for some linear function ℓ),*
- *or else f is already of class $C^{1,1}_{loc}$, in which case taking $g=f$ is the only possible choice.*

A consequence of this result is that the boundary of every compact convex body in $\mathbb{R}^n$ is of class $C^{1,1}$ up to a subset of arbitrarily small $(n-1)$-dimensional Hausdorff measure.

Corollary (Azagra-Hajłasz 2019)

Let B be a compact convex body in $\mathbb{R}^n$. Then for every $\varepsilon > 0$ there exists a convex body W_ε of class $C^{1,1}$ such that $\mathcal{H}^{n-1}(\partial B \setminus \partial W_\varepsilon) < \varepsilon$.

Sketch of the proof of the Theorem.

Consider a convex function $f : \mathbb{R}^n \to \mathbb{R}$, and assume w.l.o.g. that f is coercive. By Alexandroff's theorem we know that, for almost every $x \in \mathrm{Diff}(f)$ there exists an $n \times n$ matrix $\nabla^2 f(x)$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - \langle \nabla f(x), y - x \rangle - \frac{1}{2}\langle \nabla^2 f(x)(y-x), y - x \rangle}{|y - x|^2} = 0 \quad (\mathcal{A}_2).$$

Let $\varepsilon \in (0, 1)$ be given. Since $(\mathcal{A}_2)$ holds for almost every $x \in \mathbb{R}^n$, there exists a closed subset set $A = A_\varepsilon$ of $\mathbb{R}^n$ such that

$$A \subseteq \{x \in \mathbb{R}^n \, : \, (\mathcal{A}_2) \text{ holds at } x\},$$

and

$$\mathcal{L}^n \left(\mathbb{R}^n \setminus A \right) \leq \varepsilon/8.$$

In particular $\nabla f(x)$ exists for every $x \in A$, and by convexity the restriction of ∇f to A is continuous.

We set $B_0 = \emptyset$, and for each $k \in \mathbb{N}$, we define

$$B_k := B(0,k), \text{ and } A_k := A \cap \left(B_k \setminus B_{k-1}\right).$$

We have that

$$A = \bigcup_{k=1}^{\infty} A_k.$$

We also consider the sets

$$E_j := \{y \in A \ : f(x) - f(y) - \langle \nabla f(y), x-y \rangle \leq j|x-y|^2 \ \forall x \in \mathbb{R}^n \text{ s.t. } |x-y| \leq \frac{1}{j}\},$$

for which we have

$$A = \bigcup_{j=1}^{\infty} E_j,$$

and

$$E_j \subset E_{j+1} \text{ for all } j \in \mathbb{N}.$$

Lemma

For each $j \in \mathbb{N}$, the set E_j is closed, and in particular it is measurable.

Proof. Since A is closed, it is enough to see that E_j is closed in A. Let $(y_k)_{k\in\mathbb{N}} \subset E_j$ be such that $\lim_{k\to\infty} y_k = y \in A$, and let us check that $y \in E_j$. Given $x \in \mathbb{R}^n$ with $|x-y| < 1/j$, since $\lim_{k\to\infty} y_k = y$ there exists k_0 large enough so that $|x-y_k| < 1/j$ for all $k \geq k_0$. As $y_k \in E_j$, this implies that

$$f(x) - f(y) - \langle \nabla f(y_k), x - y_k\rangle \leq j|x-y_k|^2$$

for all $k \geq k_0$. Since the restriction of ∇f to A is continuous, hence by taking limits as $k \to \infty$ we obtain

$$f(x) - f(y) - \langle \nabla f(y), x - y\rangle \leq j|x-y|^2.$$

We have shown that this inequality holds for every y in the open ball of center x and radius $1/j$. By continuity, $y \in E_j$. □

Now, for each $k \in \mathbb{N}$, since the sequence $\{E_j\}_{j \in \mathbb{N}}$ is increasing and $A_k = \bigcup_{j=1}^{\infty} (E_j \cap A_k)$, we can find $j_k \in \mathbb{N}$ such that

$$\mathcal{L}^n(A_k \setminus E_{j_k}) \leq \frac{\varepsilon}{2^{k+3}},$$

and define, for each $k \in \mathbb{N}$,

$$C_k := E_{j_k} \cap A_k,$$

and

$$C := \bigcup_{k=1}^{\infty} C_k.$$

We may obviously assume that

$$j_k \leq j_{k+1} \text{ for all } k \in \mathbb{N}. \tag{4.2}$$

We then have that

$$\mathcal{L}^n(A \setminus C) \leq \sum_{k=1}^{\infty} \mathcal{L}^n(A_k \setminus C_k) \leq \frac{\varepsilon}{8}. \tag{4.3}$$

Lemma

For each $k \in \mathbb{N}$ there exists a number $\beta_k \geq 2$ such that:

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle \leq \beta_k |x-y|^2 \text{ for all } y \in C \cap B_k \text{ and all } x \in B_{4k}.$$

Proof. Take $y \in C_k$, and note that since (j_k) is increasing we have $C \cap B_k \subseteq E_{j_k} \cap B_k \subset B_{4k}$. If $x \in \mathbb{R}^n$ is such that $|x-y| \leq 1/j$, the inequality we seek obviously holds with $\beta_k = j_k$, because of the definition of E_{j_k}. On the other hand, if $|x-y| > 1/j_k$ and $x \in B_{4k}$, then, since f is Lipschitz on the ball B_{4k}, we have

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle \leq 2 \operatorname{Lip}\left(f_{|_{B_{4k}}} \right) |x-y| \leq 2 \operatorname{Lip}\left(f_{|_{B_{4k}}} \right) j_k |x-y|^2.$$

In any case the Lemma is satisfied with
$\beta_k = \max\left\{ 2, j_k, \, 2 j_k \operatorname{Lip}\left(f_{|_{B_{4k}}} \right) \right\}$. □

Since f is convex we have, for all $z \in C$, $x \in \mathbb{R}^n$, that

$$f(z) + \langle \nabla f(z), x - z \rangle \leq f(x),$$

hence $m(x) \leq f(x)$ for all $x \in \mathbb{R}^n$, which combined with the preceding lemma shows that the jet $(f(y), \nabla f(y))$, $y \in C$, satisfies the condition of our result for $C^{1,1}_{\mathrm{loc}}$ convex extension:

Theorem (2019)

Let C be an arbitrary nonempty subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$, $G : C \to \mathbb{R}^n$ be functions such that span$\{G(x) - G(y) : x, y \in E\} = \mathbb{R}^n$. Then there exists a convex function $F \in C^{1,1}_{loc}(\mathbb{R}^n)$ such that $F_{|C} = f$ and $(\nabla F)_{|C} = G$ if and only if for each $k \in \mathbb{N}$ there exists a number $A_k \geq 2$ such that

$$m(x) \leq f(y) + \langle G(y), x - y \rangle + \frac{A_k}{2}|x - y|^2 \quad \forall y \in E \cap B_k \;\; \forall x \in B_{4k}.$$

(We omit the proof that span$\{\nabla f(y) - \nabla f(z) : y, z \in C\} = \mathbb{R}^n$.)

End of the proof: We have thus checked that the 1-jet $(f(y), \nabla f(y))$, $y \in C$, satisfies all the conditions of the preceding Theorem, and therefore there exists a locally $C^{1,1}$ convex function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F = f$ on C, and also $\nabla F = \nabla f$ on C. In particular we have that

$$\mathcal{L}^n\left(\{x \in \mathbb{R}^n \, ; f(x) \neq F(x) \text{ or } \nabla f(x) \neq \nabla F(x)\}\right) \leq \mathcal{L}^n\left(\mathbb{R}^n \setminus C\right) \leq \varepsilon.$$

The other part of the theorem follows from the following.

Theorem (2013)

For every convex function $f : \mathbb{R}^n \to \mathbb{R}$, there exist a unique linear subspace X_f of $\mathbb{R}^n$, a unique vector $v_f \in X_f^{\perp}$, and a unique essentially coercive function $c_f : X_f \to \mathbb{R}$ such that f can be written in the form

$$f(x) = c_f(P_{X_f}(x)) + \langle v_f, x\rangle \text{ for all } x \in \mathbb{R}^n.$$

Proposition (Azagra-Hajłasz 2019)

Let $P : \mathbb{R}^n \to X$ be the orthogonal projection onto a linear subspace X of $\mathbb{R}^n$ of dimension k, with $1 \leq k \leq n-1$, let $c : X \to \mathbb{R}$ be a convex function, and define $f(x) = c(P(x))$. Then f is the only convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{L}^n(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}) < \infty$.

Proof of the Proposition. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $\mathcal{L}^n(A) < \infty$, where $A := \{x \in \mathbb{R}^n : f(x) \neq g(x)\}$. By Fubini's theorem, for $\mathcal{H}^k$-almost every point $x \in X$, we have that for $\mathcal{H}^{n-k-1}$-almost every direction $v \in X^\perp$, $|v| = 1$, the line $L(x, v) := \{x + tv : t \in \mathbb{R}\}$ must intersect A in a set of finite 1-dimensional measure. This implies that for all such $x \in X, v \in X^\perp$, the set $L(x, v) \cap (\mathbb{R}^n \setminus A)$ contains sequences

$$x_j^\pm := x + t_{x,j}^\pm v \in A, j \in \mathbb{N}$$

with $\lim_{j\to\pm\infty} t_{x,j}^\pm = \pm\infty$. Since $f = f \circ P$, this means that

$$f(x) = f(x + t_{x,j}^\pm v) = g(x + t_{x,j}^\pm v),$$

and because $t \mapsto g(x + tv)$ is convex we see that

$$f(x + tv) = f(x) = g(x + tv)$$

for all $t \in \mathbb{R}$ and every such x, v. By continuity of f and g this implies that

$$f(x + tv) = g(x + tv)$$

for all $x \in X$, $v \in X^\perp$, and this shows that $f = g$ on $\mathbb{R}^n$. □

Thank you for your attention!