

# A separable Fréchet space of almost universal disposition

Christian Bargetz

joint work with Jerzy Kąkol and Wiesław Kubiś

University of Innsbruck

Septièmes journées Besançon-Neuchâtel d'analyse fonctionnelle

20–23 June 2017



Let  $E$  and  $F$  be Banach spaces. A linear mapping

$$f: E \rightarrow F, \quad \text{with} \quad \|f(x)\|_F = \|x\|_E$$

for all  $x \in E$  is called an **isometric embedding**.



Let  $E$  and  $F$  be Banach spaces. A linear mapping

$$f: E \rightarrow F, \quad \text{with} \quad \|f(x)\|_F = \|x\|_E$$

for all  $x \in E$  is called an **isometric embedding**.

Given  $\varepsilon > 0$ , a linear mapping

$$f: E \rightarrow F, \quad \text{with} \quad (1 + \varepsilon)^{-1} \|x\|_E \leq \|f(x)\|_F \leq (1 + \varepsilon) \|x\|_E$$

for all  $x \in E$  is called an  **$\varepsilon$ -isometric embedding**.



### Theorem (Banach-Mazur, 1929)

*The space  $(C[0, 1], \|\cdot\|_\infty)$  is universal for all separable Banach spaces.*

In other words, for every separable Banach space  $E$  there is an isometric embedding  $E \hookrightarrow C[0, 1]$ .



In 1965, V. I. Gurariĭ constructed a separable Banach space with the following extension property.

- (G) For every  $\varepsilon > 0$ , for all finite dimensional normed spaces  $E \subseteq F$ , for every isometric embedding  $e: E \rightarrow \mathbb{G}$  there exists an  $\varepsilon$ -isometric embedding  $f: F \rightarrow \mathbb{G}$  such that  $f \upharpoonright E = e$ .



In 1965, V. I. Gurariĭ constructed a separable Banach space with the following extension property.

- (G) For every  $\varepsilon > 0$ , for all finite dimensional normed spaces  $E \subseteq F$ , for every isometric embedding  $e: E \rightarrow \mathbb{G}$  there exists an  $\varepsilon$ -isometric embedding  $f: F \rightarrow \mathbb{G}$  such that  $f \upharpoonright E = e$ .

In other words,

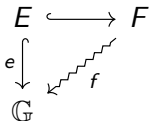
$$\begin{array}{ccc} E & \hookrightarrow & F \\ \downarrow e & & \\ \mathbb{G} & & \end{array}$$



In 1965, V. I. Gurariĭ constructed a separable Banach space with the following extension property.

(G) For every  $\varepsilon > 0$ , for all finite dimensional normed spaces  $E \subseteq F$ , for every isometric embedding  $e: E \rightarrow \mathbb{G}$  there exists an  $\varepsilon$ -isometric embedding  $f: F \rightarrow \mathbb{G}$  such that  $f \upharpoonright E = e$ .

In other words,

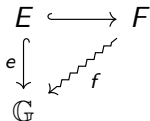




In 1965, V. I. Gurariĭ constructed a separable Banach space with the following extension property.

(G) For every  $\varepsilon > 0$ , for all finite dimensional normed spaces  $E \subseteq F$ , for every isometric embedding  $e: E \rightarrow \mathbb{G}$  there exists an  $\varepsilon$ -isometric embedding  $f: F \rightarrow \mathbb{G}$  such that  $f \upharpoonright E = e$ .

In other words,



In 1976, W. Lusky showed that (G) defines  $\mathbb{G}$  uniquely up to isometry.

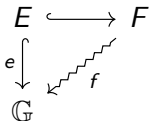




In 1965, V. I. Gurariĭ constructed a separable Banach space with the following extension property.

- (G) For every  $\varepsilon > 0$ , for all finite dimensional normed spaces  $E \subseteq F$ , for every isometric embedding  $e: E \rightarrow \mathbb{G}$  there exists an  $\varepsilon$ -isometric embedding  $f: F \rightarrow \mathbb{G}$  such that  $f \upharpoonright E = e$ .

In other words,



In 1976, W. Lusky showed that (G) defines  $\mathbb{G}$  uniquely up to isometry. A simpler proof: Kubiś and Solecki (2013)



A **Fréchet space** is a complete metrizable locally convex space.



A **Fréchet space** is a complete metrizable locally convex space.

In terms of semi-norms:

A Fréchet space is a complete topological vector space  $E$  with a **sequence** of semi-norms  $\{\|\cdot\|_i\}_{i \in \mathbb{N}}$  such that the sets

$$B_{n,\varepsilon} := \{x \in E : \max_{1 \leq i \leq n} \|x\|_i < \varepsilon\}$$

for  $\varepsilon > 0$  and  $n \in \mathbb{N}$  form a basis of neighbourhoods of zero.



A **Fréchet space** is a complete metrizable locally convex space.

In terms of semi-norms:

A Fréchet space is a complete topological vector space  $E$  with a **sequence** of semi-norms  $\{\|\cdot\|_i\}_{i \in \mathbb{N}}$  such that the sets

$$B_{n,\varepsilon} := \{x \in E : \max_{1 \leq i \leq n} \|x\|_i < \varepsilon\}$$

for  $\varepsilon > 0$  and  $n \in \mathbb{N}$  form a basis of neighbourhoods of zero.

We consider Fréchet spaces with a **fixed** sequence of semi-norms.



A **Fréchet space** is a complete metrizable locally convex space.

In terms of semi-norms:

A Fréchet space is a complete topological vector space  $E$  with a **sequence** of semi-norms  $\{\|\cdot\|_i\}_{i \in \mathbb{N}}$  such that the sets

$$B_{n,\varepsilon} := \{x \in E : \max_{1 \leq i \leq n} \|x\|_i < \varepsilon\}$$

for  $\varepsilon > 0$  and  $n \in \mathbb{N}$  form a basis of neighbourhoods of zero.

We consider Fréchet spaces with a **fixed** sequence of semi-norms.

If in addition,

$$\|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots$$

we call  $E$  a **graded Fréchet space**.



Let  $E$  and  $F$  be Fréchet spaces with fixed sequences of semi-norms.  
A linear mapping

$$f: E \rightarrow F, \quad \text{with} \quad \|f(x)\|_{F,i} = \|x\|_{E,i}$$

for all  $i \in \mathbb{N}$  and all  $x \in E$  is called an **isometric embedding**.



Let  $E$  and  $F$  be Fréchet spaces with fixed sequences of semi-norms.  
A linear mapping

$$f: E \rightarrow F, \quad \text{with} \quad \|f(x)\|_{F,i} = \|x\|_{E,i}$$

for all  $i \in \mathbb{N}$  and all  $x \in E$  is called an **isometric embedding**.

Given  $\varepsilon > 0$ , a linear mapping

$$f: E \rightarrow F, \quad \text{with} \quad (1 + \varepsilon)^{-1} \|x\|_{E,i} \leq \|f(x)\|_{F,i} \leq (1 + \varepsilon) \|x\|_{E,i}$$

for all  $i \in \mathbb{N}$  and all  $x \in E$  is called an  **$\varepsilon$ -isometric embedding**.



## Theorem (Mazur-Orlicz, 1948)

The space  $(\mathcal{C}(\mathbb{R}), \{\|\cdot\|_i\}_{i \in \mathbb{N}})$ , where

$$\|f\|_i := \sup \{|f(x)| : x \in [-i, i]\},$$

is universal for all separable Fréchet spaces.

In other words, for every separable Fréchet space  $E$  there is an isometric embedding  $E \hookrightarrow \mathcal{C}(\mathbb{R})$ .





# Is there a Fréchet-Gurariï space?

## Question

Is there an analogue of the Gurariï space for separable Fréchet spaces?



## Is there a Fréchet-Gurariï space?

### Question

Is there an analogue of the Gurariï space for separable Fréchet spaces?

A natural candidate is  $\mathbb{G}^{\mathbb{N}}$ . Can we find a suitable sequence of semi-norms on  $\mathbb{G}^{\mathbb{N}}$ ?



## Connecting to the Banach space setting

Let  $(E, \{\|\cdot\|_{i \in \mathbb{N}}\})$  be a finite dimensional graded Fréchet space.



## Connecting to the Banach space setting

Let  $(E, \{\|\cdot\|_i\}_{i \in \mathbb{N}})$  be a finite dimensional graded Fréchet space.

Observations:

- $\ker \|\cdot\|_i = \{x \in E : \|x\|_i = 0\}$  is a subspace.



Let  $(E, \{\|\cdot\|_i\}_{i \in \mathbb{N}})$  be a finite dimensional graded Fréchet space.

Observations:

- $\ker \|\cdot\|_i = \{x \in E : \|x\|_i = 0\}$  is a subspace.
- $\|\cdot\|_i$  is a norm on  $E_i := E / \ker \|\cdot\|_i$ .



Let  $(E, \{\|\cdot\|_i\}_{i \in \mathbb{N}})$  be a finite dimensional graded Fréchet space.

Observations:

- $\ker \|\cdot\|_i = \{x \in E : \|x\|_i = 0\}$  is a subspace.
- $\|\cdot\|_i$  is a norm on  $E_i := E / \ker \|\cdot\|_i$ .
- $(E_i, \|\cdot\|_i)$  is a Banach space and

$$E \hookrightarrow \prod_i E_i \quad \text{and} \quad E_{i+1} \twoheadrightarrow E_i$$



## Connecting to the Banach space setting

Let  $(E, \{\|\cdot\|_i\}_{i \in \mathbb{N}})$  be a finite dimensional graded Fréchet space.

Observations:

- $\ker \|\cdot\|_i = \{x \in E : \|x\|_i = 0\}$  is a subspace.
- $\|\cdot\|_i$  is a norm on  $E_i := E / \ker \|\cdot\|_i$ .
- $(E_i, \|\cdot\|_i)$  is a Banach space and

$$E \hookrightarrow \prod_i E_i \quad \text{and} \quad E_{i+1} \twoheadrightarrow E_i$$

- If  $f: E \rightarrow F$  is  $(\varepsilon)$ -isometric, so is  $f_i: E_i \rightarrow F_i$ , defined by

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \text{can} \downarrow & & \downarrow \text{can} \\ E_i & \xrightarrow{f_i} & F_i \end{array}$$



## An important tool: A universal operator on $\mathbb{G}$

There exists a non-expansive linear operator  $\pi: \mathbb{G} \rightarrow \mathbb{G}$  with  $\ker \pi \simeq \mathbb{G}$  and





## An important tool: A universal operator on $\mathbb{G}$

There exists a non-expansive linear operator  $\pi: \mathbb{G} \rightarrow \mathbb{G}$  with  $\ker \pi \simeq \mathbb{G}$  and

- 1 For every separable Banach space  $X$  and  $\|T\| \leq 1$ ,

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\pi} & \mathbb{G} \\ & & \uparrow T \\ & & X \end{array}$$



## An important tool: A universal operator on $\mathbb{G}$

There exists a non-expansive linear operator  $\pi: \mathbb{G} \rightarrow \mathbb{G}$  with  $\ker \pi \simeq \mathbb{G}$  and

- 1 For every separable Banach space  $X$  and  $\|T\| \leq 1$ ,  $\exists i$

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\pi} & \mathbb{G} \\ & \swarrow i & \uparrow T \\ & & X \end{array}$$



## An important tool: A universal operator on $\mathbb{G}$

There exists a non-expansive linear operator  $\pi: \mathbb{G} \rightarrow \mathbb{G}$  with  $\ker \pi \simeq \mathbb{G}$  and

- 1 For every separable Banach space  $X$  and  $\|T\| \leq 1$ ,  $\exists i$

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\pi} & \mathbb{G} \\ & \swarrow i & \uparrow T \\ & & X \end{array}$$

- 2  $\forall \varepsilon > 0$ ,  $E \subseteq F$  finite dimensional Banach spaces,  $\|T\| \leq 1$ ,

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\pi} & \mathbb{G} \\ \uparrow e & & \uparrow T \\ E & \hookrightarrow & F \end{array}$$



## An important tool: A universal operator on $\mathbb{G}$

There exists a non-expansive linear operator  $\pi: \mathbb{G} \rightarrow \mathbb{G}$  with  $\ker \pi \simeq \mathbb{G}$  and

- 1 For every separable Banach space  $X$  and  $\|T\| \leq 1$ ,  $\exists i$

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\pi} & \mathbb{G} \\ & \swarrow i & \uparrow T \\ & & X \end{array}$$

- 2  $\forall \varepsilon > 0$ ,  $E \subseteq F$  finite dimensional Banach spaces,  $\|T\| \leq 1$ ,  $\exists f$

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\pi} & \mathbb{G} \\ \uparrow e & \nearrow f & \uparrow T \\ E & \hookrightarrow & F \end{array}$$



## An important tool: A universal operator on $\mathbb{G}$

There exists a non-expansive linear operator  $\pi: \mathbb{G} \rightarrow \mathbb{G}$  with  $\ker \pi \simeq \mathbb{G}$  and

- 1 For every separable Banach space  $X$  and  $\|T\| \leq 1$ ,  $\exists i$

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\pi} & \mathbb{G} \\ & \swarrow i & \uparrow T \\ & & X \end{array}$$

- 2  $\forall \varepsilon > 0$ ,  $E \subseteq F$  finite dimensional Banach spaces,  $\|T\| \leq 1$ ,  $\exists f$

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\pi} & \mathbb{G} \\ \uparrow e & \nearrow f & \uparrow T \\ E & \hookrightarrow & F \end{array}$$

(1) implies that  $\pi$  is a projection (take  $T = \text{Id}_{\mathbb{G}}$ )



## An important tool: A universal operator on $\mathbb{G}$

There exists a non-expansive linear operator  $\pi: \mathbb{G} \rightarrow \mathbb{G}$  with  $\ker \pi \simeq \mathbb{G}$  and

- 1** For every separable Banach space  $X$  and  $\|T\| \leq 1$ ,  $\exists i$

$$\begin{array}{ccc}
 \mathbb{G} & \xrightarrow{\pi} & \mathbb{G} \\
 & \swarrow i & \uparrow T \\
 & & X
 \end{array}$$

- 2**  $\forall \varepsilon > 0$ ,  $E \subseteq F$  finite dimensional Banach spaces,  $\|T\| \leq 1$ ,  $\exists f$

$$\begin{array}{ccc}
 \mathbb{G} & \xrightarrow{\pi} & \mathbb{G} \\
 \uparrow e & \nearrow f & \uparrow T \\
 E & \hookrightarrow & F
 \end{array}$$

(1) implies that  $\pi$  is a projection (take  $T = \text{Id}_{\mathbb{G}}$ )

Based on a result by Cabello Sánchez, Garbulińska-Węgrzyn, Kubiś (2014).



## A graded sequence of semi-norms

We construct inductively a sequence of semi-norms on  $\mathbb{G}^{\mathbb{N}}$ .  
For  $x = (x_1, x_2, \dots) \in \mathbb{G}^{\mathbb{N}}$  we define

$$\|x\|_1 := \|x_1\|_{\mathbb{G}}.$$



We construct inductively a sequence of semi-norms on  $\mathbb{G}^{\mathbb{N}}$ .  
For  $x = (x_1, x_2, \dots) \in \mathbb{G}^{\mathbb{N}}$  we define

$$\|x\|_1 := \|x_1\|_{\mathbb{G}}.$$

From the properties of  $\pi$ , we may conclude that

$$\mathbb{G} = (\text{im } \pi) \oplus (\text{ker } \pi) \simeq \mathbb{G} \oplus \mathbb{G} = \mathbb{G} \times \mathbb{G}$$

holds isometrically and that there is a norm  $\|\cdot\|'_2$  on  $\mathbb{G}^2$  such that  $\mathbb{G}^2$  is isometric to  $\mathbb{G}$  and

$$\|x\|_1 = \|x_i\|_{\mathbb{G}} = \|\pi(x_1, x_2)\|_{\mathbb{G}} \leq \|(x_1, x_2)\|'_2.$$





We construct inductively a sequence of semi-norms on  $\mathbb{G}^{\mathbb{N}}$ .  
For  $x = (x_1, x_2, \dots) \in \mathbb{G}^{\mathbb{N}}$  we define

$$\|x\|_1 := \|x_1\|_{\mathbb{G}}.$$

From the properties of  $\pi$ , we may conclude that

$$\mathbb{G} = (\text{im } \pi) \oplus (\text{ker } \pi) \simeq \mathbb{G} \oplus \mathbb{G} = \mathbb{G} \times \mathbb{G}$$

holds isometrically and that there is a norm  $\|\cdot\|'_2$  on  $\mathbb{G}^2$  such that  $\mathbb{G}^2$  is isometric to  $\mathbb{G}$  and

$$\|x\|_1 = \|x_i\|_{\mathbb{G}} = \|\pi(x_1, x_2)\|_{\mathbb{G}} \leq \|(x_1, x_2)\|'_2.$$

Define  $\|x\|_2 := \|(x_1, x_2)\|'_2$  and inductively  $\|\cdot\|_n$  in a similar way.



## Proposition

*The space  $(\mathbb{G}^{\mathbb{N}}, \{\|\cdot\|_i\}_{i \in \mathbb{N}})$  is a graded Fréchet space of almost universal disposition for finite dimensional graded Fréchet spaces, i.e., for all  $\varepsilon > 0$  and for all finite dimensional graded Fréchet spaces  $E \subseteq F$  and all isometric embeddings  $f : E \rightarrow \mathbb{G}^{\mathbb{N}}$  there is an  $\varepsilon$ -isometric embedding  $g : F \rightarrow \mathbb{G}^{\mathbb{N}}$  such that  $g \upharpoonright E = f$ .*



Given  $\varepsilon > 0$ , finite dimensional graded Fréchet spaces  $E$  and  $F$  and an isomeric embedding  $f: E \rightarrow F$ , choose a sequence  $(\varepsilon_i)_{i \in \mathbb{N}}$  with  $\varepsilon_i > 0$  and

$$\prod_{i=1}^{\infty} (1 + \varepsilon_i) < 1 + \varepsilon.$$

Set

$$E_i = (E / \ker \|\cdot\|_i, \|\cdot\|_i) \quad \text{and} \quad F_i = (F / \ker \|\cdot\|_i, \|\cdot\|_i)$$

and  $f_i: E_i \rightarrow F_i$  defined by

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \text{can} \downarrow & & \downarrow \text{can} \\ E_i & \xrightarrow{f_i} & F_i. \end{array}$$

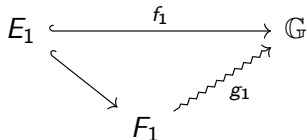


For  $i = 1$ , we have

$$\begin{array}{ccc} E_1 & \xrightarrow{f_1} & \mathbb{G} \\ & \searrow & \\ & & F_1 \end{array}$$



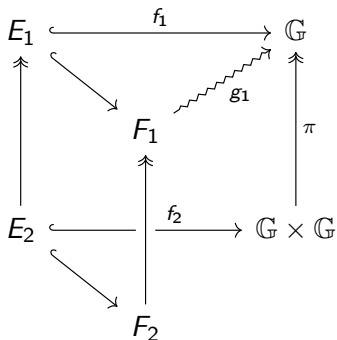
For  $i = 1$ , we have



$g_1$  is an  $\varepsilon_1$ -isometry.



For  $i = 2$ , we have

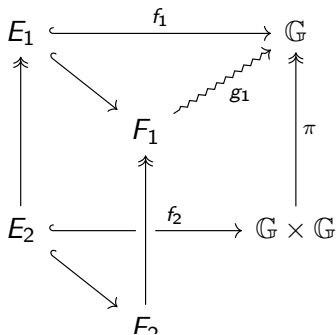


$g_1$  is an  $\varepsilon_1$ -isometry.

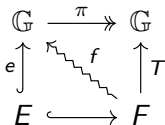


# Proof sketch II: Defining $g$ inductively

For  $i = 2$ , we have



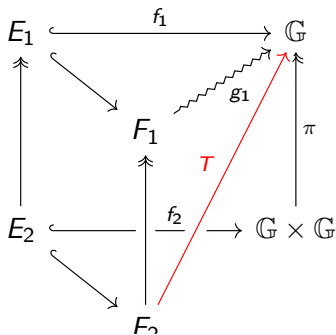
**2**  $\forall \varepsilon > 0$ ,  $E \subseteq F$  finite dimensional Banach spaces,  $\|T\| \leq 1$



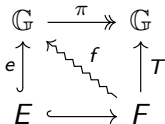


# Proof sketch II: Defining $g$ inductively

For  $i = 2$ , we have



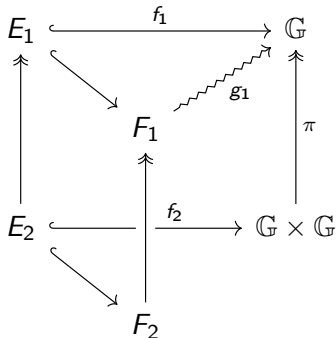
2  $\forall \varepsilon > 0$ ,  $E \subseteq F$  finite dimensional Banach spaces,  $\|T\| \leq 1$







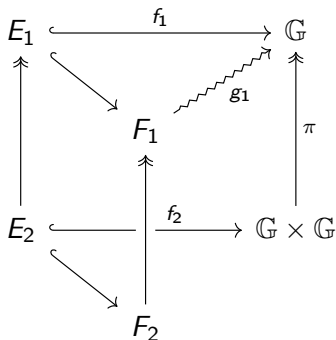
For  $i = 2$ , we have



$g_1$  is an  $\varepsilon_1$ -isometry.



For  $i = 2$ , we have



$g_1$  is an  $\varepsilon_1$ -isometry. Since  $\|g_1 \circ p_1^2\| > 1$  is possible, we need an additional step.



## Lemma

Let  $X \subset Y$  and  $A$  be finite dim. Banach spaces,  $Z$  a Banach space,  $e: X \hookrightarrow A$ ,  $T: Y \rightarrow Z$  with  $\|T\| < r$ ,  $r > 1$ , and  $\pi: A \rightarrow Z$ ,  $\|\pi\| \leq 1$ , s.t.

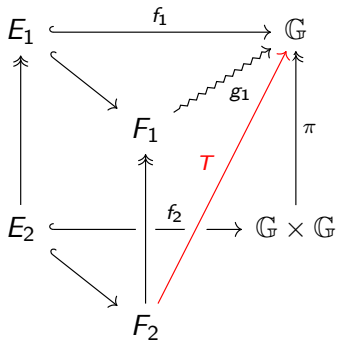
$$\begin{array}{ccc} A & \xrightarrow{\pi} & Z \\ e \uparrow & & \uparrow T \\ X & \hookrightarrow & Y \end{array}$$

There exists a finite dim. Banach space  $C$ ,  $i_A: A \hookrightarrow C$ , an  $(r-1)$ -isometric embedding  $i_Y: Y \rightarrow C$  and  $\pi': C \rightarrow Z$ ,  $\|\pi'\| \leq 1$  s.t. we get the commutative diagram

$$\begin{array}{ccccc} & & & & Z \\ & & & \nearrow \pi & \uparrow T \\ A & \xrightarrow{i_A} & C & \xrightarrow{\pi'} & Z \\ e \uparrow & & \nwarrow i_Y & & \\ X & \hookrightarrow & Y & & \end{array}$$

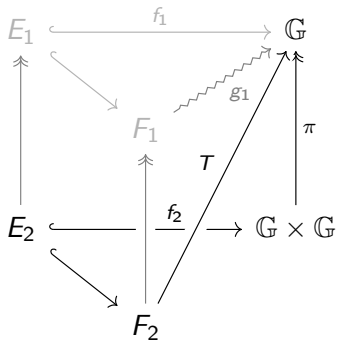


## Proof sketch III: The additional step



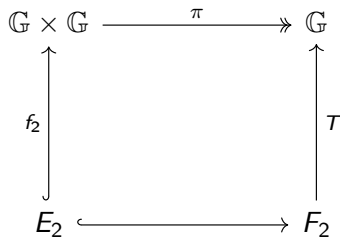
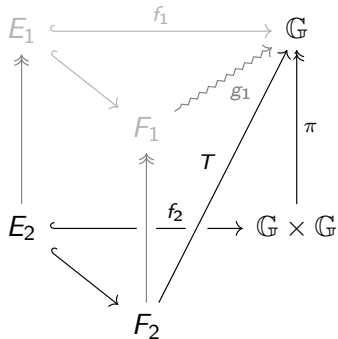


## Proof sketch III: The additional step



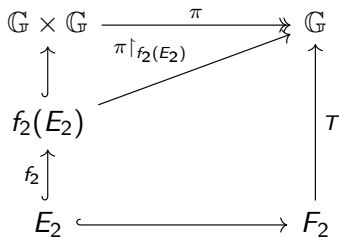
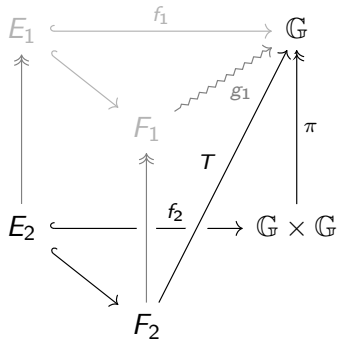


# Proof sketch III: The additional step



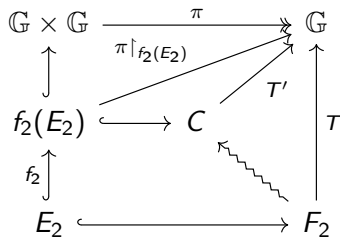
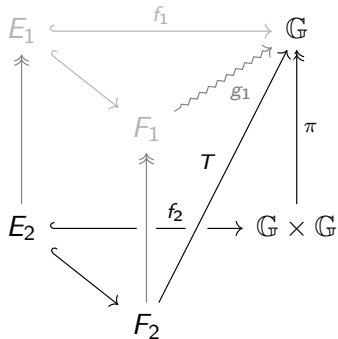


# Proof sketch III: The additional step





# Proof sketch III: The additional step

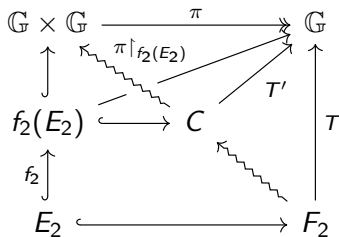
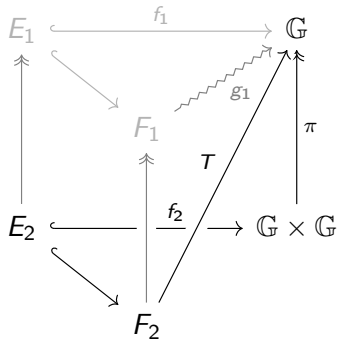


$$\|T'\| \leq 1$$



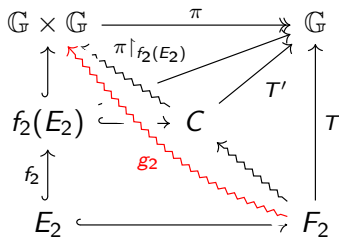
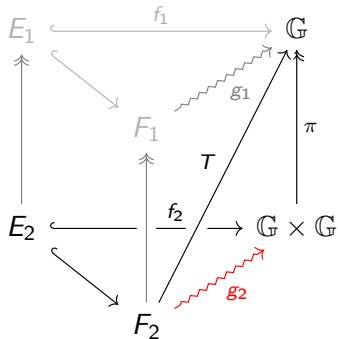


# Proof sketch III: The additional step





# Proof sketch III: The additional step

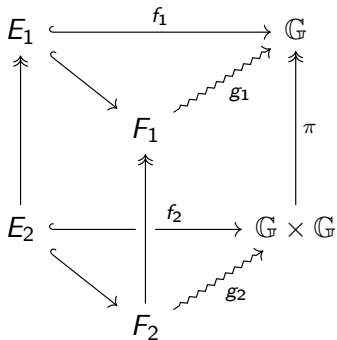


The map  $g_2$  satisfies

$$(1 + \varepsilon_2)^{-1}(1 + \varepsilon_1)^{-1} \|x\|_2 \leq \|g_2(x)\|_2 \leq (1 + \varepsilon_2)(1 + \varepsilon_1) \|x\|_2.$$

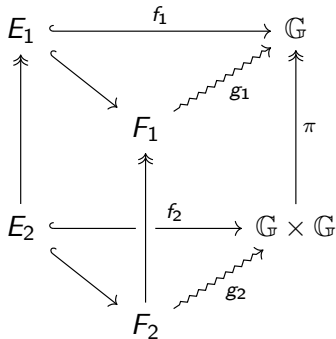


## Proof sketch III: The additional step





## Proof sketch III: The additional step



Then we continue inductively.



- The space  $\mathbb{G}^{\mathbb{N}}$  is unique up to isometries.



- The space  $\mathbb{G}^{\mathbb{N}}$  is unique up to isometries.
- It is universal for separable (graded) Fréchet spaces.




- The space  $\mathbb{G}^{\mathbb{N}}$  is unique up to isometries.
- It is universal for separable (graded) Fréchet spaces.
- There is no separable Fréchet space of universal disposition (where the extension of isometric embeddings would be an isometric embedding and not just  $\varepsilon$ -isometric).



- The space  $\mathbb{G}^{\mathbb{N}}$  is unique up to isometries.
- It is universal for separable (graded) Fréchet spaces.
- There is no separable Fréchet space of universal disposition (where the extension of isometric embeddings would be an isometric embedding and not just  $\varepsilon$ -isometric).
- No space of the form  $\mathcal{C}(X)$ , where  $X$  is a hemi-compact topological space, is of almost universal disposition for finite dimensional (graded) Fréchet spaces.





-  Christian Bargetz, Jerzy Kąkol, and Wiesław Kubiś.  
A separable Fréchet space of almost universal disposition.  
Preprint, 2016.
-  Félix Cabello Sánchez, Joanna Garbulińska-Węgrzyn, and Wiesław Kubiś.  
Quasi-Banach spaces of almost universal disposition.  
*J. Funct. Anal.*, 267(3):744–771, 2014.
-  V. I. Gurariĭ.  
Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces.  
*Sibirsk. Mat. Ž.*, 7:1002–1013, 1966.
-  Wiesław Kubiś and Sławomir Solecki.  
A proof of uniqueness of the Gurariĭ space.  
*Israel J. Math.*, 195(1):449–456, 2013.