A separable Fréchet space of almost universal disposition

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joint work with Jerzy Kąkol and Wiesław Kubiś

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Notation

Let E and F be Banach spaces. A linear mapping

$$f\colon E \to F, \qquad \text{with} \qquad \|f(x)\|_F = \|x\|_E$$

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for all $x \in E$ is called an ε-isometric embedding.

Theorem (Banach-Mazur, 1929)

The space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$ *is universal for all separable Banach spaces.*

In other words, for every separable Banach space E there is an isometric embedding $E \hookrightarrow \mathcal{C}[0,1]$.

The Gurariĭ space

In 1965, V. I. Gurariĭ constructed a separable Banach space with the following extension property.

(G) For every $\varepsilon > 0$, for all finite dimensional normed spaces $E \subseteq F$, for every isometric embedding $e\colon E \to \mathbb{G}$ there exists an ε-isometric embedding $f\colon F \to \mathbb{G}$ such that $f \restriction E = e$.

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In 1976, W. Lusky showed that (G) defines $\mathbb{G}$ uniquely up to isometry. A simpler proof: Kubiś and Solecki (2013)

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In terms of semi-norms:
A Fréchet space is a complete topological vector space E with a sequence of semi-norms $\{\|\cdot\|_i\}_{i\in\mathbb{N}}$ such that the sets

$$B_{n,\varepsilon} := \{x \in E\colon \max_{1\leq i\leq n} \|x\|_i < \varepsilon\}$$

for $\varepsilon > 0$ and $n \in \mathbb{N}$ form a basis of neighbourhoods of zero.

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We consider Fréchet spaces with a fixed sequence of semi-norms.

If in addition,

$$\|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots$$

we call E a graded Fréchet space.

Let E and F be Fréchet spaces with fixed sequences of semi-norms. A linear mapping

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A universal Fréchet space

Theorem (Mazur-Orlicz, 1948)

The space $(\mathcal{C}(\mathbb{R}), \{\|\cdot\|_i\}_{i\in\mathbb{N}})$, *where*

$$\|f\|_i := \sup\left\{|f(x)| : x \in [-i, i]\right\},$$

is universal for all separable Fréchet spaces.

In other words, for every separable Fréchet space E there is an isometric embedding $E \hookrightarrow \mathcal{C}(\mathbb{R})$.

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Is there an analogue of the Gurariĭ space for separable Fréchet spaces?

A natural candidate is $\mathbb{G}^{\mathbb{N}}$. Can we find a suitable sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$?

Connecting to the Banach space setting

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$$E \hookrightarrow \prod_i E_i \qquad \text{and} \qquad E_{i+1} \twoheadrightarrow E_i$$

- If $f\colon E \to F$ is (ε-)isometric, so is $f_i\colon E_i \to F_i$, defined by

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "can"'] & F \arrow[d, "can"] \\ E_i \arrow[r, "f_i"] & F_i \end{tikzcd}$$

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1. For every separable Banach space X and $\|T\| \leq 1$,

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2. $\forall \varepsilon > 0$, $E \subseteq F$ finite dimensional Banach spaces, $\|T\| \leq 1$,

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Based on a result by Cabello Sánchez, Garbulińska-Węgrzyn, Kubiś (2014).

A graded sequence of semi-norms

We construct inductively a sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$.
For $x = (x_1, x_2, \ldots) \in \mathbb{G}^{\mathbb{N}}$ we define

$$\|x\|_1 := \|x_1\|_{\mathbb{G}}.$$

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From the properties of π, we may conclude that

$$\mathbb{G} = (\operatorname{im}\pi) \oplus (\ker\pi) \simeq \mathbb{G} \oplus \mathbb{G} = \mathbb{G} \times \mathbb{G}$$

holds isometrically and that there is a norm $\|\cdot\|_2'$ on $\mathbb{G}^2$ such that $\mathbb{G}^2$ is isometric to $\mathbb{G}$ and

$$\|x\|_1 = \|x_i\|_{\mathbb{G}} = \|\pi(x_1, x_2)\|_{\mathbb{G}} \leq \|(x_1, x_2)\|_2'.$$

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Define $\|x\|_2 := \|(x_1, x_2)\|_2'$ and inductively $\| \cdot \|_n$ in a similar way.

$\mathbb{G}^{\mathbb{N}}$ is of almost universal disposition

Proposition

The space $(\mathbb{G}^{\mathbb{N}}, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ *is a graded Fréchet space of almost universal disposition for finite dimensional graded Fréchet spaces, i.e., for all* $\varepsilon > 0$ *and for all finite dimensional graded Fréchet spaces* $E \subseteq F$ *and all isometric embeddings* $f\colon E \to \mathbb{G}^{\mathbb{N}}$ *there is an* ε*-isometric embedding* $g\colon F \to \mathbb{G}^{\mathbb{N}}$ *such that* $g \restriction E = f$.

Proof sketch

Given $\varepsilon > 0$, finite dimensional graded Fréchet spaces E and F and an isomeric embedding $f\colon E \to F$, choose a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ with $\varepsilon_i > 0$ and

$$\prod_{i=1}^{\infty}(1+\varepsilon_i) < 1+\varepsilon.$$

Set

$$E_i = (E/\ker\|\cdot\|_i, \|\cdot\|_i) \qquad \text{and} \qquad F_i = (F/\ker\|\cdot\|_i, \|\cdot\|_i)$$

and $f_i\colon E_i \to F_i$ defined by

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ E_i & \xrightarrow{f_i} & F_i. \end{array}$$

Proof sketch II: Defining g inductively

For $i = 1$, we have

$$\begin{tikzcd} E_1 \arrow[rr, hook, "f_1"] \arrow[dr, hook] & & \mathbb{G} \\ & F_1 & \end{tikzcd}$$

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g_1 is an ε_1-isometry.

For $i = 2$, we have

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E_2 \arrow[uu, two heads] \arrow[rr, hook, "f_2" near end] \arrow[dr, hook] & & \mathbb{G} \times \mathbb{G} \arrow[uu, two heads, "\pi"'] \\
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$$\begin{tikzcd}
\mathbb{G} \arrow[r, two heads, "\pi"] & \mathbb{G} \\
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& F_2 \arrow[uu, two heads, crossing over] &
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\end{tikzcd}$$

g_1 is an ε_1-isometry. Since $\left\| g_1 \circ p_1^2 \right\| > 1$ is possible, we need an additional step.

Lemma

Let $X \subset Y$ and A be finite dim. Banach spaces, Z a Banach space, $e\colon X \hookrightarrow A$, $T\colon Y \to Z$ with $\|T\| < r$, $r > 1$, and $\pi\colon A \to Z$, $\|\pi\| \leq 1$, s.t.

$$\begin{tikzcd}
A \arrow[r, "\pi"] & Z \\
X \arrow[u, hook, "e"] \arrow[r, hook] & Y \arrow[u, "T"']
\end{tikzcd}$$

There exists a finite dim. Banach space C, $i_A\colon A \hookrightarrow C$, an $(r-1)$-isometric embedding $i_Y\colon Y \to C$ and $\pi'\colon C \to Z$, $\|\pi'\| \leq 1$ s.t. we get the commutative diagram

$$\begin{tikzcd}
 & & Z \\
A \arrow[urr, "\pi"] \arrow[r, hook, "i_A"'] & C \arrow[ur, "\pi'"'] & \\
X \arrow[u, hook, "e"] \arrow[rr, hook] & & Y \arrow[uu, "T"'] \arrow[ul, rightsquigarrow, "i_Y"]
\end{tikzcd}$$

Proof sketch III: The additional step

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\begin{tikzcd}
E_1 \arrow[rr, hook, "f_1"] \arrow[dr, hook] & & \mathbb{G} \\
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\mathbb{G} \times \mathbb{G} \arrow[rr, two heads, "\pi"] & & \mathbb{G} \\
f_2(E_2) \arrow[u, hook] \arrow[urr, "\pi\restriction_{f_2(E_2)}"] & & \\
E_2 \arrow[u, hook, "f_2"] \arrow[rr, hook] & & F_2 \arrow[uu, "T"']
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$$\|T'\| \leq 1$$

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\begin{tikzcd}
\mathbb{G} \times \mathbb{G} \arrow[rr, two heads, "\pi"] & & \mathbb{G} \\
f_2(E_2) \arrow[u, hook] \arrow[r, hook] \arrow[urr, "\pi\restriction_{f_2(E_2)}"] & C \arrow[ul, rightsquigarrow] \arrow[ur, "T'"'] & \\
E_2 \arrow[u, hook, "f_2"] \arrow[rr, hook] & & F_2 \arrow[uu, "T"'] \arrow[ul, rightsquigarrow]
\end{tikzcd}
$$

Proof sketch III: The additional step

$$
\begin{tikzcd}
E_1 \arrow[rr, hook, "f_1"] \arrow[dr, hook] & & \mathbb{G} \\
& F_1 \arrow[ur, squiggly, "g_1"'] & \\
E_2 \arrow[uu, two heads] \arrow[rr, hook, "f_2"] \arrow[dr, hook] & & \mathbb{G} \times \mathbb{G} \arrow[uu, two heads, "\pi"'] \\
& F_2 \arrow[uu, two heads] \arrow[uuur, "T"] \arrow[ur, squiggly, "g_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{G} \times \mathbb{G} \arrow[rr, two heads, "\pi"] & & \mathbb{G} \\
f_2(E_2) \arrow[u, hook] \arrow[r, hook, dashed] & C \arrow[ul, squiggly, dashed] \arrow[ur, "\pi\restriction_{f_2(E_2)}"] \arrow[ur, "T'"'] & \\
E_2 \arrow[u, "f_2"] \arrow[rr, hook] & & F_2 \arrow[uu, "T"'] \arrow[ul, squiggly] \arrow[uull, squiggly, "g_2"]
\end{tikzcd}
$$

The map g_2 satisfies

$$(1+\varepsilon_2)^{-1}(1+\varepsilon_1)^{-1}\|x\|_2 \leq \|g_2(x)\|_2 \leq (1+\varepsilon_2)(1+\varepsilon_1)\|x\|_2.$$

Proof sketch III: The additional step

$$
\begin{tikzcd}
E_1 \arrow[rr, hook, "f_1"] \arrow[dr, hook] & & \mathbb{G} \\
& F_1 \arrow[ur, squiggly, "g_1"'] & \\
E_2 \arrow[uu, two heads] \arrow[rr, hook, "f_2"] \arrow[dr, hook] & & \mathbb{G} \times \mathbb{G} \arrow[uu, two heads, "\pi"'] \\
& F_2 \arrow[uu, two heads] \arrow[ur, squiggly, "g_2"'] &
\end{tikzcd}
$$

Proof sketch III: The additional step

$$
\begin{tikzcd}
E_1 \arrow[rr, hook, "f_1"] \arrow[dr, hook] & & \mathbb{G} \\
& F_1 \arrow[ur, rightsquigarrow, "g_1"'] & \\
E_2 \arrow[uu, two heads] \arrow[rr, hook, "f_2"] \arrow[dr, hook] & & \mathbb{G} \times \mathbb{G} \arrow[uu, two heads, "\pi"'] \\
& F_2 \arrow[uu, two heads] \arrow[ur, rightsquigarrow, "g_2"'] &
\end{tikzcd}
$$

Then we continue inductively.

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Properties of $\mathbb{G}^{\mathbb{N}}$ / Comparison to the Banach space case

- The space $\mathbb{G}^{\mathbb{N}}$ is unique up to isometries.
- It is universal for separable (graded) Fréchet spaces.
- There is no separable Fréchet space of universal disposition (where the extension of isometric embeddings would be an isometric embedding and not just ε-isometric).

Properties of $\mathbb{G}^{\mathbb{N}}$ / Comparison to the Banach space case

- The space $\mathbb{G}^{\mathbb{N}}$ is unique up to isometries.
- It is universal for separable (graded) Fréchet spaces.
- There is no separable Fréchet space of universal disposition (where the extension of isometric embeddings would be an isometric embedding and not just ε-isometric).
- No space of the form $\mathcal{C}(X)$, where X is a hemi-compact topological space, is of almost universal disposition for finite dimensional (graded) Fréchet spaces.

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