Some Best Proximity Point Theorems in G- Metric Spaces

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Overview

Abstract

When a non-self mapping has no fixed points, it could be interesting to study the existence and uniqueness of some points that minimize the distance between an origin and its corresponding image. These points are known as best proximity points and they were introduced by Fan [Mathematische Zeitschrift, vol. 112, no. 3 pp. 234-240, 1969]. Interestingly, best proximity point theorems also serve as a natural generalization of fixed point theorems, a best proximity point becomes a fixed point if the mapping under consideration is a self-mapping. Study of this kind of points and their properties has become one of the newest branches of fixed point theory, and many interesting results, generalizing the notion of fixed point, have been presented.

Very recently, N. Hussain et al.[Abstract and Applied Analysis, vol. 2014, art. ID 837943] introduced certain new class of proximal contraction mappings and established the best proximity point theorem in G- metric spaces. In this note, acknowledging the aforesaid concept, some best proximity point theorems are proved under generalized cyclic contraction condition which is new for this setting in the frame work of G- metric spaces.

Suitable examples are also presented which substantiate the genuineness of our investigations in this note.

Introduction

It is evident that the fixed point theory is one of the fundamental tools in nonlinear functional analysis. The celebrated Banach contraction mapping principle [3] is the most known and crucial result in fixed point theory. It says that each contraction in a complete metric space has a unique fixed point. This theorem not only guarantees the existence and uniqueness of the fixed point but also shows how to evaluate this point. By virtue of this fact, the Banach contraction mapping principle has been generalized in many ways over the years. ([[4]-[7]]).
The Banach Contraction Principle states that, if a self-mapping T of a complete metric space X is a contraction mapping, then T has a unique fixed point.
In 2003 Kirk-Srinavasan-Veeramani [8] introduced the notion of cyclic contraction mapping and proved some fixed point theorems for the operators in the class of cyclic contraction. In 2005, Eldred, Kirk and Veeramani [9] proved the existence of a best proximity point for relatively non-expansive mappings by using the notion of proximal normal structure. In 2006, Eldred and Veeramani [10] introduced the notion of cyclic contraction and gave sufficient condition for the existence of a best proximity point for a cyclic contraction mapping T on a uniformly convex Banach space. Fixed point theory plays an important role in furnishing a uniform treatment to solve various equations of the form $Tx = x$ for self-mappings T defined on subsets of metric spaces.

Given two nonempty subsets A and B of a metric space, consider a non-self mapping T from A to B. Because T is not a self-mapping, the equation $Tx = x$ is unlikely to have a solution. Therefore, it is of primary importance to seek an element x that in some sense is closest to Tx. That is, when the equation $Tx = x$ has no solution, one tries to determine an approximate solution x subject to the condition that the distance between x and Tx is minimal. Best approximation theorems and best proximity point theorems are relevant in this perspective.

A classical best approximation theorem, due to Fan [1], states that if A is a nonempty compact and convex subset of a Hausdorff locally convex topological vector space X and $T : A \to X$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. There have been many subsequent extensions and variants of Fans Theorem, see [[11], [12], [13],[14], [15],[16]] and references therein. On the other hand, though best approximation theorems ensure the existence of approximate solutions, such results need not yield optimal solutions. But, best proximity point theorems furnish sufficient conditions that assure the existence of approximate solutions which are optimal as well. Indeed, if there is no exact solution to the fixed point equation $Tx = x$ for a non-self mapping $T : A \to B$, then a best proximity theorem offers sufficient conditions for the existence of an optimal approximate solution x, called a best proximity point of the mapping T , satisfying the condition that $d(x, Tx) = d(A, B)$. A best proximity point theorem for non-self proximal contractions has been investigated in [17].

In the case of cyclic contractive mapping $T : A \cup B \to A \cup B$, a point $x \in A \cup B$ is called the best proximity point if $d(x, Tx) = d(A, B)$. Notice that a best proximity point x is a fixed point of T whenever $A \cap B \neq \phi$. Thus it generalizes the notion of fixed point in case when $A \cap B = \phi$. Further [18], [19] , [20], [21], [22],[23] examine several variants of contractions for the existence of a best proximity point.

On the other hand, in 2004, Mustafa and Sims[24],[25] introduced a new generalized metric space structure and called it, G-metric space. The authors also portrayed some fixed point theorems in perspective of G- metric spaces [26],[27]. Tagging on these initial papers, several researchers established many fixed point results on the setting of G- metric spaces.[28],[29],[30], [31].

Preliminaries

Consistent with Mustafa and Sims[24],[25], the following definitions and results will be needed in the sequel.

Definition ([25])

Let X be a nonempty set, and let $G : X \times X \times X \to R^{+}$ be a function satisfying the following properties:

(G-1) $G(x, y, z) = 0$ if $x = y = z$.

(G-2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$.

(G-3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$

(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, symmetry in all three variables,

(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ The function G is called a generalized or a G-metric on X and the pair (X, G) is called a G-metric space.

Definition ([25])

Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, therefore we say that the sequence $\{x_n\}$ is G-convergent to $x \in X$ if $\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0$, that is for any $\epsilon > 0$, there exists $N \in \mathcal{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $m, n > N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n,m \to +\infty} x_n = x$.

Proposition[25] Let (X, G) be a G-metric space. Then the followings are equivalent:

- $\{x_n\}$ is G-convergent to x;
- $G(x_n, x_n, x) \to 0$ as $n \to +\infty$;
- $G(x_n, x, x) \to 0$ as $n \to +\infty$;
- $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$;

Definition ([25])

Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathcal{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$ that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition[25] Let (X, G) be a G-metric space. Then the followings are equivalent:

- $\{x_n\}$ is G-Cauchy;
- for every $\epsilon > 0$, there is $N \in \mathcal{N}$, $G(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq N$;

Definition ([25])

A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition ([25])

Let (X, G) be a G-metric space. A mapping $F : X \times X \times X \to X$ is said to be continuous if, for any three G- convergent sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converging to x, y and z, respectively, $\{F(x_n, y_n, z_n)\}$ is $G-$ convergent to $F(x, y, z)$.

Lemma[25] By the rectangle inequality (G-5) together with the symmetry (G-4), we have

$$\begin{aligned} G(x, y, y) = G(y, y, x) &\leq G(y, x, x) + G(x, y, x) \\ &= 2G(y, x, x). \end{aligned}$$

Every G-metric on X defines a metric d_G on X given by

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \qquad \text{for all } x,y \in X$$

Very recently N. Hussain [2] introduced best proximity point concept in G- metric spaces and established the best proximity point theorems for new class of proximal contraction mappings. N. Hussain [2], defined the followings:
Let (X, G) be a $G-$ metric space. Suppose that A and B are non empty subset of a G metric space (X, G). Then

$$\begin{aligned} A_0 &= \{x \in A : d_G(x,y) = d_G(A,B) \ \ for \ \ some \ \ y \in B\}, \\ B_0 &= \{y \in B : d_G(x,y) = d_G(A,B) \ \ for \ \ some \ \ x \in A\}, \end{aligned} \tag{1}$$

where $d_G(A,B) = inf\{d_G(x,y) : x \in A, y \in B\}$.
If $T : A \cup B \to A \cup B$ is a cyclic contractive mapping in a G- metric space (X, G), a point $x \in A \cup B$ is called the best proximity point of T if $d_G(x, Tx) = d_G(A, B)$ or equivalently we can say $G(x, Tx, Tx) = G(A, B, B)$ and $G(x, x, Tx) = G(A, A, B)$. Where
$G(A, B, B) = \inf\{G(a, b, b) : a \in A, b \in B\}$
and $G(A, A, B) = \inf\{G(a, a, b) : a \in A, b \in B\}$.

Definition ([2])

Let (X, G) be a $G-$ metric space and let A and B be two non empty subsets of X. Then B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ in B, satisfying the condition $d_G(x, y_n) \to d_G(x, B)$ for some x in A, has a convergent sub-sequence.

Main Results

First of all generalized proximal cyclic weak ϕ contractive mapping is defined in the framework of G- metric spaces.

Definition

Let A and B be two non-empty sub sets of G metric space (X, G). Let $T : A \cup B \to A \cup B$ be such that $T(A) \subseteq B, T(B) \subseteq A$. We say that T is generalized proximal cyclic weak ϕ contractive mapping if, for

$$x, u, u^* \in A \ \ and \ \ v, y \in B,$$

$$d_G(u^*, Tx) = d_G(A, B);$$

$$d_G(u, Tu^*) = d_G(A, B);$$

$$d_G(v, Ty) = d_G(A, B).$$

$$\Rightarrow G(u^*, u, v) \leq M(x, v, y) - \phi(M(x, v, y)), \tag{2}$$

where $M(x, v, y) = \max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$ and $\phi \in \Phi$, the set of continuous function $\phi : [0, \infty] \to [0, \infty]$ with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$.

Example

Let $X = R$. Define $G : X \times X \times X \to R^+$ as

$$G(x,y,z) = \frac{1}{2}\max\{|x-y|, |y-z|, |z-x|\}$$

then (X, G) is a G-metric space and $d_G(x,y) = |x-y|$. Define $A = \{0,2,4\}, B = \{1,3,5\}$. Let $T : A \cup B \to A \cup B$ defined by

$$Tx = \begin{cases} 0 & if \;\; x = 5 \;; \\ x+1 & otherwise, \end{cases}$$

clearly $T(A) \subseteq B$ and $T(B) \subseteq A, d_G(A,B) = 1$ then T is a generalized proximal cyclic weak ϕ cyclic contraction for $u = 2, u^* = 2, x = 0 \in A$ and $v = 1, y = 5 \in B$.
We have

$$d_G(u^*, Tx) = d_G(A,B);$$

$$d_G(u, Tu^*) = d_G(A,B);$$

$$d_G(v, Ty) = d_G(A,B).$$

$$\Rightarrow G(u^*, u, v) \leq M(x,v,y) - \phi(M(x,v,y)),$$

where $M(x,v,y) = max\{G(x,v,y), G(x, Tx, Tx), G(y, Ty, Ty)\}$ and choosing $\phi(t) = \frac{t}{2}$.

Following theorem is proved for generalized proximal cyclic weak ϕ contraction mappings.

Theorem

Let A, B be a two non-empty subsets of a G metric space (X, G) such that (A, G) is complete G-metric space, A_0 is non-empty and B is approximatively compact with respect to A.
Assume that $T : A \cup B \to A \cup B$ is a generalized proximal cyclic weak ϕ contraction mapping such that $T(A) \subseteq B, T(B) \subseteq A$. Then T has a best proximity point.

Proof: First of all we construct a sequence of Picard iteration as usual. Define the sequence $\{x_n\}$ as $x_n = Tx_{n-1}$, n= 1,2,3, ...
It is given that $A_0 \subseteq A$ is non empty then we have $x_0 \in A_0 \subseteq A$. Since T is cyclic so

$$x_1 = Tx_o \in T(A_0) \subseteq B_0 \subset B.$$

Then

$$d_G(x_0, Tx_0) = d_G(A, B)$$

or

$$d_G(x_0, x_1) = d_G(A, B)$$

here $x_1 \in B$ then we must have

$$x_2 = Tx_1 \in T(B_0) \subset A$$

such that

$$d_G(x_1, Tx_1) = d_G(A, B)$$

or

$$d_G(x_1, x_2) = d_G(A, B)$$

Recursively, we obtain a sequence $\{x_n\}$ in $A_0 \cup B_0 \subseteq A \cup B$ satisfying $d_G(x_n, x_{n+1}) = d_G(A, B)$, n= 0,1,2,3, . . .
This shows that

$$d_G(u^*, Tx) = d_G(A, B);$$

$$d_G(u, Tu^*) = d_G(A, B);$$

$$d_G(v, Ty) = d_G(A, B),$$

with $u^* = x_{n+1}, x = x_{n-1}, u = x_{n+1}, y = x_n, v = x_n$. Therefore from (2) we obtain

$$G(x_{n+1}, x_{n+1}, x_n) \leq M(x_{n-1}, x_n, x_n) - \phi(M(x_{n-1}, x_n, x_n)), \quad (3)$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n) &= max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, Tx_n, Tx_n), G(x_n, Tx_n, Tx_n)\} \\ &= max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\ &= max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

Now, if

$$M(x_{n-1}, x_n, x_n) = G(x_n, x_{n+1}, x_{n+1}),$$

then from 3

$$G(x_n, x_{n+1}, x_{n+1}) = G(x_n, x_{n+1}, x_{n+1}) - \phi(G(x_n, x_{n+1}, x_{n+1}))$$

$$\Rightarrow \phi(G(x_n, x_{n+1}, x_{n+1})) = 0$$

$$\Rightarrow G(x_n, x_{n+1}, x_{n+1})) = 0$$

$$\Rightarrow x_n = x_{n+1},$$

which is not true, if for any n_0, $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} become fixed point of T. Then we have

$$M(x_{n-1}, x_n, x_n) = G(x_{n-1}, x_n, x_n).$$

Therefore

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq G(x_{n-1}, x_n, x_n) - \phi(G(x_{n-1}, x_n, x_n)) \\ &\leq G(x_{n-1}, x_n, x_n). \end{aligned} \tag{4}$$

Thus sequence $G(x_n, x_{n+1}, x_{n+1})$ is a non-negative, non-increasing sequence which converge to $L \geq 0$.
Letting $n \to \infty$ in (4), we obtain

$$L \leq L - \phi(L)$$

$\Rightarrow \phi(L) = 0 \Rightarrow L = 0 \quad i.e.$

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{5}$$

Now we claim that $\{x_n\}$ is a G-Cauchy sequence in (X, G). On the contrary we assume that $\{x_n\}$ is not G-Cauchy. Then there exist an $\epsilon > 0$ and corresponding sub-sequences $\{n(k)\}$ and $\{m(k)\}$ of N satisfying $n(k) > m(k) > k$ for which

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \epsilon, \tag{6}$$

where $n(k)$ is chosen as the smallest integer satisfying (6), i.e.

$$G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) < \epsilon. \tag{7}$$

It is easy to conclude from (6) and (7) and the rectangle inequality, that

$$\begin{aligned} \epsilon &\leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\leq G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \epsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}). \end{aligned} \tag{8}$$

Taking limit $k \to \infty$ in (8) and utilizing (5), we obtain

$$\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon. \tag{9}$$

Observe that for every $k \in N$, there exist $s(k)$ satisfying $0 \leq s(k) \leq m$ such that

$$n(k) - m(k) + s(k) \equiv 1(m). \tag{10}$$

Therefore large enough values of k we have $r(k) = m(k) - s(k) > 0$ and $x_{r(k)}$ and $x_{n(k)}$ lie in the set A and B respectively.
Next using (2) with $u^* = x_{r(k)}, u = x_{n(k)+1}, v = x_{n(k)}, x = x_{r(k)}, y = x_{n(k)}$, we obtain

$$G(x_{r(k)}, x_{n(k)+1}, x_{n(k)}) \leq M(x_{r(k)}, x_{n(k)}, x_{n(k)}) - \phi(M(x_{r(k)}, x_{n(k)}, x_{n(k)})), \tag{11}$$

where

$$\begin{aligned} &M(x_{r(k)}, x_{n(k)}, x_{n(k)}) \\ &= max\{G(x_{r(k)}, x_{n(k)}, x_{n(k)}), G(x_{r(k)}, Tx_{r(k)}, Tx_{r(k)}), G(x_{n(k)}, Tx_{n(k)}, Tx_{r(k)})\} \\ &= max\{G(x_{r(k)}, x_{n(k)}, x_{n(k)}), G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}), G(x_{n(k)}, x_{n(k)+1}, Tx_{r(k)+1}.\} \end{aligned}$$

Employing rectangle inequality repeatedly, we observe that

$$\begin{aligned}
G(x_{r(k)}, x_{n(k)}, x_{n(k)}) \\
&\leq G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}) + G(x_{r(k)+1}, x_{n(k)}, x_{n(k)}) \\
&\leq G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}) + G(x_{r(k)+1}, x_{r(k)+2}, x_{r(k)+2}) + G(x_{r(k)+2}, x_{n(k)}, x_{n(k)}) \\
&\vdots \\
&\leq \sum_{i=r}^{m-1} [G(x_{i(k)}, x_{i(k)+1}, x_{i(k)+1})] + G(x_{m(k)}, x_{n(k)}, x_{n(k)})
\end{aligned}$$

or equivalently

$$0 \leq G(x_{r(k)}, x_{n(k)}, x_{n(k)}) - G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq \sum_{i=r}^{m-1} G(x_{i(k)}, x_{i(k)+1}, x_{i(k)+1}). \tag{12}$$

Notice that the sum on the R.H.S. of (12) contains $s - 1 \leq M(finite)$ number of terms, and due to (5) each term of this sum tends to 0 as $k \to \infty$. Therefore,

$$\lim_{n\to\infty} G(x_{r(k)}, x_{n(k)}, x_{n(k)}) = \lim_{n\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon \tag{13}$$

using rectangle inequality again, we have

$$\begin{aligned}
0 &\leq G(x_{r(k)}, x_{n(k)+1}, x_{n(k)}) \\
&\leq G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)}).
\end{aligned} \tag{14}$$

On letting $k \to \infty$ and using (13), we deduce that

$$\lim_{k\to\infty} G(x_{r(k)}, x_{n(k)+1}, x_{n(k)}) = \epsilon \tag{15}$$

Now passing to limit as $k \to \infty$ in (11) and using (5),(13),(15), we get

$$\begin{aligned}\epsilon &\leq max\{\epsilon, 0, 0\} - \phi(max\{\epsilon, 0, 0\}),\\ &= \epsilon - \phi(\epsilon)\end{aligned}$$

and hence $\phi(\epsilon) = 0 \Rightarrow \epsilon = 0$, which contradicts the assumption that $\{x_n\}$ is not G-Cauchy. Thus $\{x_n\}_0^\infty$ is a Cauchy sequence.
Since A and B is complete there exist $z \in A \subseteq A \cup B$ such that $x_n \to z$ as $n \to \infty$.
On the other hand $\forall n \in N$, we can write

$$\begin{aligned} d_G(z, B) &\leq d_G(z, Tx_n) = d_G(z, x_{n+1})\\ &\leq d_G(z, x_n) + d_G(x_n, x_{n+1}) \\ &\leq d_G(z, x_n) + d_G(A, B).\end{aligned} \tag{16}$$

Taking limit as $n \to \infty$ in (16), we get

$$d_G(z, B) \leq d_G(A, B)$$

but

$$d_G(A, B) \leq d_G(z, B), forz \in A.$$

So

$$\lim_{n\to\infty} d_G(z, Tx_n) = d_G(z, B) = d_G(A, B). \tag{17}$$

Since B is approximatively compact with respect to A, so the sequence $\{Tx_n\}$ has a sub-sequence $\{Tx_{n(K)}\}$ that converge to some $p \in B \subseteq A \cup B$. Hence

$$\begin{aligned} d_G(z, p) &= \lim_{n\to\infty} d_G(x_{n(k)}, Tx_{n(k)}) \\ &= \lim_{n\to\infty} d_G(x_{n(k)}, x_{n(k)+1}) = d_G(A, B) \\ & i.e. \;\; d_G(z, p) = d_G(A, B). \end{aligned} \tag{18}$$

So $z \in A_0$. Now since $T(z) \in T(A_0) \subseteq B_0$, $\quad \exists \quad w \in A_0$ such that

$$d_G(w, Tz) = d_G(A, B)$$

Now we claim that $w = z$. For our assertion utilizing (2) with $x = x_{n-1}, y = x_n, v = x_n, u = w, u^* = z$, we have

$$\begin{aligned} G(z, w, x_n) &\leq M(x_{n-1}, x_n, x_n) - \phi(M(x_{n-1}, x_n, x_n)) \\ &\leq max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\ &- \phi(max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}). \end{aligned}$$

Letting $n \to \infty$ we have

$$G(z,z,w) \leq G(z,z,z) - \phi(G(z,z,z))$$
$$\Rightarrow (G(z,z,w)) = 0 \Rightarrow z = w$$

thus

$$d_G(z, Tz) = d_G(A, B). \tag{19}$$

Therefore T has the best proximity point.
Following example substantiates the hypothesis of aforesaid theorem:

Example

Let $X = R$ and $G : X \times X \times X \to R^+$ defined by

$$G(x,y,z) = \frac{1}{2}\max\{|x-y|, |y-z|, |z-x|\}$$

then clearly (X, G) is a G-metric space.
Now $d_G(x,y) = |x-y|$.
Let $A = \{0,-1,-2,-3,-4\}$ and $B = \{1,2,3,4\}$.
Define $T : A \cup B \to A \cup B$ by

$$T(x) = \begin{cases} 1 & if \ x = -4 \\ -x+1 & , otherwise. \end{cases}$$

Clearly $T(A) \subseteq B$ and $T(B) \subseteq A$.
Also Taking $\phi(t) = \frac{t}{2}$.

Example (Continued)

Clearly $d_G(A, B) = 1$.
Here $A_0 = \{0\}$.
Now if we choose $u = 0, u^* = 0, x = -4 \in A$ and $v = 1, y = 1 \in B$
Then

$$d_G(u^*, Tx) = d_G(A, B),$$
$$d_G(u, Tu^*) = d_G(A, B),$$

and

$$d_G(v, Ty) = d_G(A, B)$$

Now with $u = u^* = 0, x = -4,\ v = y = 1$, we verify the Condition (2).

$$G(u^*, u, v) = G(0, 0, 1) = \frac{1}{2}.$$

Now

$$\begin{aligned} M(x, v, y) &= \max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\} \\ &= \max\{G(-4, 1, 1), G(-4, 1, 1), G(1, 0, 0)\} = \frac{5}{2}. \end{aligned}$$

and

$$M(x, v, y) - \phi(M(x, v, y)) = \frac{5}{2} - \frac{1}{2}(\frac{5}{2}) = \frac{5}{4}.$$

Hence

$$G(u^*, u, v) = \frac{1}{2} \leq \frac{5}{4} = M(x, v, y) - \phi(M(x, v, y)).$$

That is

$$d_G(u^*, Tx) = d_G(A, B),$$
$$d_G(u, Tu^*) = d_G(A, B),$$

and

$$d_G(v, Ty) = d_G(A, B)$$

implies that

$$G(u^*, u, v) \leq M(x, v, y) - \phi(M(x, v, y)),$$

where

$$M(x, v, y) = \max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}.$$

Thus T is a generalized weak ϕ proximal cyclic mapping.
All the conditions of Theorem(9) are satisfied and T has a best proximity point $z = 0$.
Since $d_G(0, T0) = d_G(A, B)$.

For particular choices of functions ϕ we obtain following corollaries.

Corollary

Let A, B be a two non-empty subsets of a G metric space (X, G) such that (A, G) is complete G-metric space, A_0 is non-empty and B is approximatively compact with respect to A.
Assume that $T : A \cup B \to A \cup B$ satisfying

(i) $T(A) \subseteq B, T(B) \subseteq A$.

(ii) If for $u^*, u, x \in A$ and $y, v \in B$.

$$d_G(u^*, Tx) = d_G(A, B);$$

$$d_G(u, Tu^*) = d_G(A, B);$$

$$d_G(v, Ty) = d_G(A, B).$$

$$\Downarrow$$

$$G(u^*, u, v) \leq kM(x, v, y),$$

where $M(x, v, y) = max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$ and $k \in (0, 1)$.
Then T has a best proximity point.

Proof: The proof is obvious by taking $\phi(t) = (1 - k)t, k \in (0, 1)$.

Corollary

Let A, B be a two non-empty subsets of a G metric space (X, G) such that (A, G) is complete G-metric space, A_0 is non-empty and B is approximatively compact with respect to A.
Assume that $T : A \cup B \to A \cup B$ satisfying

(i) $T(A) \subseteq B, T(B) \subseteq A$

(ii) If for $u^*, x, u \in A$ and $y, v \in B$

$$d_G(u^*, Tx) = d_G(A, B);$$

$$d_G(u, Tu^*) = d_G(A, B);$$

$$d_G(v, Ty) = d_G(A, B),$$

$$\Rightarrow G(u^*, u, v) \leq \alpha G(x, v, y) + \beta G(x, Tx, Tx) + \gamma G(y, Ty, Ty),$$

where $x, u^*, u \in A, \quad v, y \in B$ and $\alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma < 1$. Then T has a best proximity point.

Proof: Clearly we have

$$\alpha G(x, v, y) + \beta G(x, Tx, Tx) + \gamma G(y, Ty, Ty) \leq (\alpha + \beta + \gamma) M(x, v, y).$$

Thus

$$G(u^*, u, v) \leq (\alpha + \beta + \gamma) M(x, v, y).$$

Using corollary 12, with $k = (\alpha + \beta + \gamma) \in (0, 1)$, we obtain that T has a best proximity point.
Next corollary is obtained for cyclic mapping satisfying integral type contractive conditions.

Corollary

Let A, B be a two non-empty subsets of a G metric space (X, G) such that (A, G) is complete G-metric space, A_0 is non-empty and B is approximatively compact with respect to A.
Assume that $T : A \cup B \to A \cup B$ satisfying

(i) $T(A) \subseteq B, T(B) \subseteq A$.

(ii) If for $u^*, u, x \in A$ and $y, v \in B$,

$$d_G(u^*, Tx) = d_G(A, B); \qquad d_G(u, Tu^*) = d_G(A, B); \qquad d_G(v, Ty) = d_G(A, B).$$

$$\Rightarrow \int_o^{G(u*,u,v)} ds \leq \int_o^{M(x,v,y)} ds - \phi(\int_o^{M(x,v,y)} ds),$$

where $\phi \in \Phi$ and

$$M(x, v, y) = max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\},$$

where $x, u^*, u \in A, \quad v, y \in B$. Then T has a best proximity point.

Utilizing corollary 13, we obtain next corollary.

Corollary

Let A, B be a two non-empty subsets of a G metric space (X, G) such that (A, G) is complete G-metric space, A_0 is non-empty and B is approximatively compact with respect to A.
Assume that $T : A \cup B \to A \cup B$ satisfying

(i) $T(A) \subseteq B, T(B) \subseteq A$.

(ii) If for $u^*, x, u \in A$ and $y, v \in B$,

$$d_G(u^*, Tx) = d_G(A, B); \quad d_G(u, Tu^*) = d_G(A, B); \quad d_G(v, Ty) = d_G(A, B).$$

$$\Rightarrow \int_o^{G(u*,u,v)} ds \leq k \int_o^{M(x,v,y)} ds,$$

where $k \in (0, 1)$ and

$$M(x, v, y) = max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$$

Then T has a best proximity point.

Best proximity point via generalized proximal cyclic contraction

In this section, the generalized proximal cyclic contraction is defined and the existence of best proximity point is established in the framework of G-metric spaces.

Definition

Let (X, G) be a G-metric space. A map $T : A \cup B \to A \cup B$ is called a generalized cyclic proximity contraction if following conditions hold:
(1) $T(A) \subseteq B$ and $T(B) \subseteq A$.
(2)

$$G(Tx, Ty, Tz) \leq a_1 G(x, y, z) + a_2 G(x, Tx, Tx) + a_3 G(y, Ty, Ty) + a_4 G(z, Tz, Tz) + [1 - (a_1 + a_2 + a_3 + a_4)] G(A, B, B). \tag{20}$$

Where $a_1, a_2, a_3, a_4 \geq 0$ and $a_1 + a_2 + a_3 + a_4 < 1$ for all $x \in A$ and $y, z \in B$.
Or

$$G(Tx, Ty, Tz) \leq a_1 G(x, y, z) + a_2 G(x, x, Tx) + a_3 G(y, y, Ty) + a_4 G(z, z, Tz) + [1 - (a_1 + a_2 + a_3 + a_4)] G(A, A, B). \tag{21}$$

Where $a_1, a_2, a_3, a_4 \geq 0$ and $a_1 + a_2 + a_3 + a_4 < 1$ for all $x \in A$ and $y, z \in B$
$G(A, B, B) = \inf\{G(a, b, b) : a \in A, b \in B\}$ and $G(A, A, B) = \inf\{G(a, a, b) : a \in A, b \in B\}$.

Example

Let $X = R$. Consider G-metric space (X, G) defined by

$$G(x, y, z) = \frac{1}{2}\max\{|x-y|, |y-z|, |z-x|\}$$

Suppose $A = [-1, 0)$ and $B = (0, 1]$ and $T : A \cup B \to A \cup B$ is defined by $Tx = \frac{-x}{3}$
then T is generalized cyclic proximity contraction mapping with $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{10}$, $a_3 = \frac{1}{10}$, $a_4 = \frac{1}{5}$.

Lemma Let A and B be non-empty subsets of a G-metric space (X, G). Suppose that the mapping $T : A \cup B \to A \cup B$ be a generalized cyclic proximity contraction. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = G(A, B, B)$$

or

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = G(A, A, B)$$

Proof:- Suppose $x_0 \in A \cup B$. Define a sequence $\{x_n\}$ by $x_n = Tx_{n-1}, \forall n \in N$.
Now from (20), we obtain

$$\begin{aligned} G(x_1, x_2, x_2) &= G(Tx_0, Tx_1, Tx_1) \\ &\leq a_1 G(x_0, x_1, x_1) + a_2 G(x_0, Tx_0, Tx_0) + a_3 G(x_1, Tx_1, Tx_1) + a_4 G(x_1, Tx_1, Tx_1) + \\ &\quad [1-(a_1+a_2+a_3+a_4)]G(A, B, B) \\ &= a_1 G(x_0, x_1, x_1) + a_2 G(x_0, x_1, x_1) + a_3 G(x_1, x_2, x_2) + a_4 G(x_1, x_2, x_2) + \\ &\quad [1-(a_1+a_2+a_3+a_4)]G(A, B, B). \end{aligned}$$

Therefore

$$G(x_1, x_2, x_2) = \frac{a_1 + a_2}{1 - a_3 - a_4} G(x_0, x_1, x_1) + \left(1 - \frac{a_1 + a_2}{1 - a_3 - a_4}\right) G(A, B, B).$$

This implies that

$$\begin{aligned} G(x_1, x_2, x_2) - G(A, B, B) &= \frac{a_1 + a_2}{1 - a_3 - a_4}[G(x_0, x_1, x_1) - G(A, B, B)] \\ &= \lambda(G(x_0, x_1, x_1) - G(A, B, B)). \end{aligned} \tag{22}$$

Where $\lambda = \frac{a_1+a_2}{1-a_3-a_4} < 1$.
Now

$$\begin{aligned} G(x_2, x_3, x_3) &= G(Tx_1, Tx_2, Tx_2) \\ &\leq a_1 G(x_1, x_2, x_2) + a_2 G(x_1, Tx_1, Tx_1) + a_3 G(x_2, Tx_2, Tx_2) + a_4 G(x_2, Tx_2, Tx_2) + \\ &\quad [1-(a_1+a_2+a_3+a_4)]G(A, B, B) \end{aligned}$$

$$G(x_2, x_3, x_3) = \frac{a_1 + a_2}{1 - a_3 - a_4} G(x_1, x_2, x_2) + \left(1 - \frac{a_1 + a_2}{1 - a_3 - a_4}\right) G(A, B, B)$$

$$\Rightarrow G(x_2, x_3, x_3) - G(A, B, B) \le \lambda[G(x_1, x_2, x_2) - G(A, B, B)]$$

Inductively, we obtain

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) - G(A, B, B) &\le \lambda[G(x_{n-1}, x_n, x_n) - G(A, B, B)] \\ &\le \lambda.\lambda[G(x_{n-2}, x_{n-1}, x_{n-1}) - G(A, B, B)] \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots \\ &\le \lambda^n[G(x_0, x_1, x_1) - G(A, B, B)], \end{aligned}$$

i.e.

$$G(x_n, x_{n+1}, x_{n+1}) - G(A, B, B) \le \lambda^n[G(x_0, x_1, x_1) - G(A, B, B)]. \tag{23}$$

Since $\lambda < 1$, and so $\lim_{n \to \infty} \lambda^n = 0$.

Now taking $n \to \infty$ in (23), we get

$$\lim_{n \to \infty} G(x_n, x_{n+1}, x_{n+1}) = G(A, B, B). \tag{24}$$

Applying similar argument as above for (21), we arrive at

$$\lim_{n \to \infty} G(x_n, x_{n+1}, x_{n+1}) = G(A, A, B). \tag{25}$$

Our main theorem of this section runs as follows:

Theorem

Let A and B be non-empty subsets of a G-metric space (X, G). Suppose that the mapping $T : A \cup B \to A \cup B$ be a generalized cyclic contraction between A and B. If $x_n = Tx_{n-1}$ and the sequence $\{x_n\}$has a sub sequence converging to some element x in A. Then x is a best proximity point of T.

Proof:: Since $T : A \cup B \to A \cup B$ is a generalized cyclic proximity contraction. Now suppose $\{x_{n_k}\}$ be a any sub sequence of $\{x_n\}$ converging to x in $A \subseteq A \cup B$.
Then by Lemma(29), we have

$$\lim_{k\to\infty} G(x_{n_k-1}, x_{n_k}, x_{n_k}) = G(A, B, B).$$

Further, Consider

$$\begin{aligned} G(x_{n_k}, Tx, Tx) &= G(Tx_{n_k-1}, Tx, Tx) \\ &\leq a_1 G(x_{n_k-1}, x, x) + a_2 G(x_{n_k-1}, Tx_{n_k-1}, Tx_{n_k-1}) + a_3 G(x, Tx, Tx) + \\ &\quad a_4 G(x, Tx, Tx) + [1 - (a_1 + a_2 + a_3 + a_4)] G(A, B, B) \\ &= a_1 G(x_{n_k-1}, x, x) + a_2 G(x_{n_k-1}, x_{n_k}, x_{n_k}) + a_3 G(x, Tx, Tx) + \\ &\quad a_4 G(x, Tx, Tx) + [1 - (a_1 + a_2 + a_3 + a_4)] G(A, B, B) \\ &\leq a_1 [G(x_{n_k-1}, x_{n_k}, x_{n_k}) + G(x_{n_k}, x, x)] + a_2 G(x_{n_k-1}, x_{n_k}, x_{n_k}) + \\ &\quad (a_3 + a_4)[G(x, x_{n_k}, x_{n_k}) + G(x_{n_k}, Tx, Tx)] + [1 - (a_1 + a_2 + a_3 + a_4)] G(A, B, B). \end{aligned}$$

This implies that

$$G(x_{n_k}, Tx, Tx) \leq \frac{a_1 + a_2}{1 - a_3 - a_4} G(x_{n_k - 1}, x_{n_k}, x_{n_k}) + \frac{a_1 + 2a_3 + 2a_4}{1 - a_3 - a_4} G(x_{n_k}, x, x) + \left(1 - \frac{a_1 + a_2}{1 - a_3 - a_4}\right) G(A, B, B) \tag{26}$$

$$\Rightarrow G(x_{n_k}, Tx, Tx) \leq \frac{a_1 + a_2}{1 - a_3 - a_4} G(x_{n_k - 1}, x_{n_k}, x_{n_k}) + \frac{a_1}{1 - a_3 - a_4} G(x_{n_k}, x, x) + \frac{a_3 + a_4}{1 - a_3 - a_4} G(x_{n_k}, x_{n_k}, x) + \left(1 - \frac{a_1 + a_2}{1 - a_3 - a_4}\right) G(A, B, B)$$

Since $\{x_{n_k}\}$ converges to x as then we conclude that

$$\lim_{k \to \infty} G(x_{n_k}, Tx, Tx) = G(A, B, B).$$

Therefore

$$G(x, Tx, Tx) = G(A, B, B) \tag{27}$$

Imposing similar argument on condition(21) with (25), finally we arrive at

$$G(x, x, Tx) = G(A, A, B) \tag{28}$$

Now from (27) and (28), we obtain

$$d_G(x, Tx) = d_G(A, B)$$

This implies that x is a best proximity point of T. This completes the proof.

Now an example is presented which illustrates the validity of above theorem.

Example

Let $X = R$. Define $G : X \times X \times X \to R^+$ by

$$G(x, y, z) = \frac{1}{2} \max\{|x - y|, |y - z|, |z - x|\}.$$

Then (X, G) is a G-metric space. Let $A = [0, 1]$ and $B = [2, 3]$.
Define $T : A \cup B \to A \cup B$ as

$$T(x) = \begin{cases} 2, & \text{if } x \in [0, 1], \\ 1, & \text{if } x \in [2, 3]. \end{cases}$$

Then clearly $T(A) \subseteq B$ and $T(B) \subseteq A$. Applying routine calculation, one can show that T satisfies following condition for all $x \in A$ and $y, z \in B$ and for for $a_1 = 0.1, a_2 = 0.3 = a_3$ and $a_4 = 0.2$.

$$G(Tx, Ty, Tz) \leq a_1 G(x, y, z) + a_2 G(x.Tx, Tx) + a_3 G(y, Ty, Ty) + a_4 G(z, Tz, Tz) + (1 - (a_1 + a_2 + a_3 + a_4))G(A, B, B)$$

Thus T is a generalized proximal cyclic contraction mapping.
Now there exists a sequence $\{x_n\} = \{1 - \frac{1}{n}\}$, $n = 1, 2, 3...$, which has a convergent sub-sequence, converging to $x = 1 \in A$.
Thus T satisfied all the conditions of Theorem9. Hence T has a proximal point $x = 1$ such that $d_G(x, Tx) = d_G(A, B) = 1$.

Remarks:

1. If we take $a_1 = 0$, $a_2 = a_3 = a_4 = k$ then Condition(21) reduces to Kannan type proximal cyclic contraction for $k \in (0, \frac{1}{3})$.

2. If we chose $a_1 = a_2 = a_3 = a_4 = k$ then Condition(21) reduces to Reich type proximal cyclic contraction for $k \in (0, \frac{1}{4})$.

3. If we take $a_1 + a_2 + a_3 + a_4 = k$ and $M(x, y, z) = max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}$ then Condition(21) reduces to Ciric type proximal cyclic contraction for $k \in (0, 1)$.

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Thank You

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