

# On a conjecture of Pisier

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Typical examples :

Gauss and Poisson semigroups  $e^{t\Delta}$  and  $e^{-t(-\Delta)^{1/2}}$  on  $\mathbb{T}$  or on  $\mathbb{R}^n$ .

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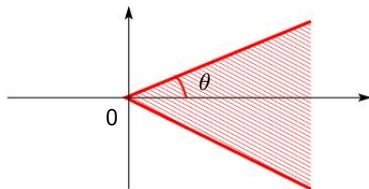
$$\widehat{T(f)} = \varphi \cdot \widehat{f}, \quad f \in L^p(G) \cap L^2(G).$$

# Bounded analytic semigroups

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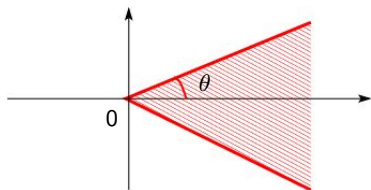
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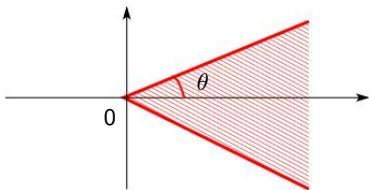
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Let  $(T_t)_{t \geq 0}$  be a  $C_0$ -semigroup on a Banach space  $X$ .

We say that it is a bounded analytic semigroup if  $(T_t)_{t > 0}$  admits a bounded analytic extension

$$\begin{array}{ccc} \Sigma_\theta & \longrightarrow & B(X) \\ z & \longmapsto & T_z \end{array}$$

for some  $0 < \theta < \frac{\pi}{2}$ .



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- $T_t(1) = 1$  for any  $t \geq 0$  (unital, mass conservation)

are bounded analytic on  $L^p(\Omega)$ .

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$$\begin{aligned} P: L^2(\Omega) &\longrightarrow L^2(\Omega) \\ f &\longmapsto \sum_{k=1}^{\infty} \left( \int_{\Omega} f \varepsilon_k \right) \varepsilon_k \end{aligned}$$

induces a bounded operator on the Bochner space  $L^2(\Omega, X)$ .



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When we replace  $L^p$ -spaces by Bochner spaces, the semigroup remains bounded analytic.

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The scalar case  $X = \mathbb{C}$  is the result of Stein.

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$X$   $K$ -convex  $\longleftrightarrow$   $E$   $OK$ -convex  $\iff S^p(E)$  is  $K$ -convex.

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# A partial answer

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- The latter problem is part of a more general framework :  
a similar question exists for other Banach spaces properties  
(UMD, cotype...).



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# Schur multipliers

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A Schur multiplier defined by a matrix  $A$  is a map

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Recall that

$$S^p = \left\{ x \in B(\ell^2) : \|x\|_{S^p} = (\operatorname{Tr} |x|^p)^{\frac{1}{p}} < \infty \right\}$$

where  $|x| = (x^*x)^{\frac{1}{2}}$ .

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## Theorem (C. A., 2014)

The  $w^*$ -semigroups of contractive Schur multipliers on  $B(\ell^2)$  which are selfadjoint on  $S^2$  are precisely the semigroups

$$T_t = \left[ e^{-t \|\alpha_i - \beta_j\|_H^2} \right]_{i,j \geq 1},$$

where  $\alpha_i, \beta_j$  are elements of a real Hilbert space  $H$ .

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The regularity of a semigroup is, roughly spoken, induced by the quality of the approximation of the identity operator as the parameter goes to zero.

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- $N_1 \supset N_2 \supset \dots$  is a decreasing sequence of subalgebras of  $N$  associated to conditional expectations  $\mathbb{E}_k$ .

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$$\left\| \sum_{i,j} e_{ij} \otimes e_{ij} \otimes \cdots \otimes e_{ij} \otimes x_{ij} \right\|_{SP(B(\ell^2)^{\otimes n}, E)} = \left\| \sum_{i,j} e_{ij} \otimes x_{ij} \right\|_{SP(E)} .$$



It was a pleasure to present this work to you !