On a conjecture of Pisier

Cédric Arhancet

University of Franche-Comté - France

Besançon - October 2014

Symmetric diffusion semigroup

Definition (E. M. Stein, ≈1970)

Let Ω be a measure space.

Symmetric diffusion semigroup

Definition (E. M. Stein, ≈1970)

Let Ω be a measure space.
Let $(T_t)_{t \geq 0}$ be a w^*-semigroup of contractive operators on $L^\infty(\Omega)$.

Symmetric diffusion semigroup

Definition (E. M. Stein, ≈1970)

Let Ω be a measure space.
Let $(T_t)_{t \geq 0}$ be a w^*-semigroup of contractive operators on $L^\infty(\Omega)$.

We say that $(T_t)_{t \geq 0}$ is a symmetric diffusion semigroup if

Symmetric diffusion semigroup

Definition (E. M. Stein, ≈1970)

Let Ω be a measure space.
Let $(T_t)_{t\geq 0}$ be a w^*-semigroup of contractive operators on $L^\infty(\Omega)$.

We say that $(T_t)_{t\geq 0}$ is a symmetric diffusion semigroup if

- *it induces a C_0-semigroup $(T_t)_{t\geq 0}$ of contractions on $L^p(\Omega)$ for any $1 \leq p < \infty$ (contractiveness),*

Symmetric diffusion semigroup

Definition (E. M. Stein, ≈1970)

Let Ω be a measure space.
Let $(T_t)_{t \geq 0}$ be a w^*-semigroup of contractive operators on $L^\infty(\Omega)$.

We say that $(T_t)_{t \geq 0}$ is a symmetric diffusion semigroup if

- *it induces a C_0-semigroup $(T_t)_{t \geq 0}$ of contractions on $L^p(\Omega)$ for any $1 \leq p < \infty$ (contractiveness),*
- *each T_t is selfadjoint on $L^2(\Omega)$ (symmetry).*

Symmetric diffusion semigroup

Definition (E. M. Stein, ≈1970)

Let Ω be a measure space.
Let $(T_t)_{t\geq 0}$ be a w^*-semigroup of contractive operators on $L^\infty(\Omega)$.

We say that $(T_t)_{t\geq 0}$ is a symmetric diffusion semigroup if

- *it induces a C_0-semigroup $(T_t)_{t\geq 0}$ of contractions on $L^p(\Omega)$ for any $1 \leq p < \infty$ (contractiveness),*
- *each T_t is selfadjoint on $L^2(\Omega)$ (symmetry).*

Typical examples :
Gauss and Poisson semigroups $e^{t\Delta}$ and $e^{-t(-\Delta)^{1/2}}$ on $\mathbb{T}$ or on $\mathbb{R}^n$.

Fourier multipliers

- A Fourier multiplier on $\mathbb{T}$ defined by a sequence of complex numbers (φ_k) of $\ell^\infty(\mathbb{Z})$ is an operator

Fourier multipliers

- A Fourier multiplier on $\mathbb{T}$ defined by a sequence of complex numbers (φ_k) of $\ell^\infty(\mathbb{Z})$ is an operator

$$T: \begin{array}{ccc} L^p(\mathbb{T}) & \longrightarrow & L^p(\mathbb{T}) \\ \sum_k x_k e^{2i\pi k\cdot} & \longmapsto & \sum_k \varphi_k x_k e^{2i\pi k\cdot}. \end{array}$$

Fourier multipliers

- A Fourier multiplier on $\mathbb{T}$ defined by a sequence of complex numbers (φ_k) of $\ell^\infty(\mathbb{Z})$ is an operator

$$T: \begin{array}{ccc} L^p(\mathbb{T}) & \longrightarrow & L^p(\mathbb{T}) \\ \sum_k x_k e^{2i\pi k\cdot} & \longmapsto & \sum_k \varphi_k x_k e^{2i\pi k\cdot}. \end{array}$$

- Let G be an abelian locally compact group, $\widehat{G}$ be its dual group and let $\widehat{\cdot}$ be the Fourier transform.

Fourier multipliers

- A Fourier multiplier on $\mathbb{T}$ defined by a sequence of complex numbers (φ_k) of $\ell^\infty(\mathbb{Z})$ is an operator

$$\begin{array}{rccc} T\colon & L^p(\mathbb{T}) & \longrightarrow & L^p(\mathbb{T}) \\ & \sum_k x_k e^{2i\pi k\cdot} & \longmapsto & \sum_k \varphi_k x_k e^{2i\pi k\cdot}. \end{array}$$

- Let G be an abelian locally compact group, $\widehat{G}$ be its dual group and let $\widehat{\cdot}$ be the Fourier transform.
  More generally, we say that $T\colon L^p(G) \to L^p(G)$ is a Fourier multiplier if there exists a

Fourier multipliers

- A Fourier multiplier on $\mathbb{T}$ defined by a sequence of complex numbers (φ_k) of $\ell^\infty(\mathbb{Z})$ is an operator

$$T\colon \begin{array}{ccc} L^p(\mathbb{T}) & \longrightarrow & L^p(\mathbb{T}) \\ \sum_k x_k e^{2i\pi k\cdot} & \longmapsto & \sum_k \varphi_k x_k e^{2i\pi k\cdot} \end{array}.$$

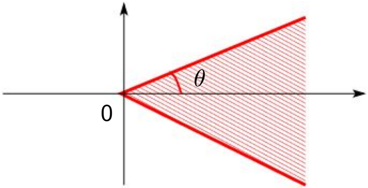
- Let G be an abelian locally compact group, $\widehat{G}$ be its dual group and let $\widehat{.}$ be the Fourier transform.
More generally, we say that $T\colon L^p(G) \to L^p(G)$ is a Fourier multiplier if there exists a function $\varphi \in L^\infty(\widehat{G})$ such that

$$\widehat{T(f)} = \varphi \cdot \widehat{f}, \qquad f \in L^p(G) \cap L^2(G).$$

Bounded analytic semigroups

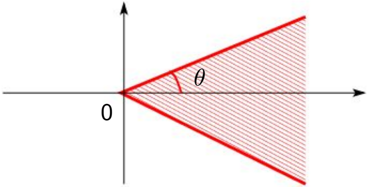
For any $0 < \theta < \pi$

For any $0 < \theta < \pi$, we let $\Sigma_\theta = \{z \in \mathbb{C}^* : -\theta < \mathrm{Arg}(z) < \theta\}$ the open sector of angle 2θ around the positive real axis $(0, \infty)$.



Bounded analytic semigroups

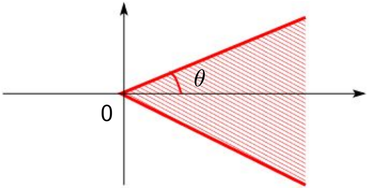
For any $0 < \theta < \pi$, we let $\Sigma_\theta = \{z \in \mathbb{C}^* : -\theta < \mathrm{Arg}(z) < \theta\}$ the open sector of angle 2θ around the positive real axis $(0, \infty)$.



Let $(T_t)_{t \geq 0}$ be a C_0-semigroup on a Banach space X.

Bounded analytic semigroups

For any $0 < \theta < \pi$, we let $\Sigma_\theta = \{z \in \mathbb{C}^* : -\theta < \mathrm{Arg}(z) < \theta\}$ the open sector of angle 2θ around the positive real axis $(0, \infty)$.



Let $(T_t)_{t \geq 0}$ be a C_0-semigroup on a Banach space X.

We say that it is a bounded analytic semigroup if $(T_t)_{t>0}$ admits a bounded analytic extension

$$\begin{array}{ccc} \Sigma_\theta & \longrightarrow & B(X) \\ z & \longmapsto & T_z \end{array}$$

for some $0 < \theta < \frac{\pi}{2}$.

Theorem (E. M. Stein, 1970)

Let Ω be a measure space.

Bounded analytic semigroups

Theorem (E. M. Stein, 1970)

Let Ω be a measure space.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup on $L^\infty(\Omega)$.

Bounded analytic semigroups

Theorem (E. M. Stein, 1970)

Let Ω be a measure space.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup on $L^\infty(\Omega)$.
Then $(T_t)_{t \geq 0}$ induces a bounded analytic semigroup on $L^p(\Omega)$ for any $1 < p < \infty$.

Theorem (E. M. Stein, 1970)

Let Ω be a measure space.
Let $(T_t)_{t\geq 0}$ be a symmetric diffusion semigroup on $L^\infty(\Omega)$.
Then $(T_t)_{t\geq 0}$ induces a bounded analytic semigroup on $L^p(\Omega)$ for any $1 < p < \infty$.

Particular case :
the symmetric diffusion semigroups $(T_t)_{t\geq 0}$ such that

Bounded analytic semigroups

Theorem (E. M. Stein, 1970)

Let Ω be a measure space.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup on $L^\infty(\Omega)$.
Then $(T_t)_{t \geq 0}$ induces a bounded analytic semigroup on $L^p(\Omega)$ for any $1 < p < \infty$.

Particular case :
the symmetric diffusion semigroups $(T_t)_{t \geq 0}$ such that

- $T_t(f) \geq 0$ if $f \geq 0$ (positivity),

Bounded analytic semigroups

Theorem (E. M. Stein, 1970)

Let Ω be a measure space.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup on $L^\infty(\Omega)$.
Then $(T_t)_{t \geq 0}$ induces a bounded analytic semigroup on $L^p(\Omega)$ for any $1 < p < \infty$.

Particular case :
the symmetric diffusion semigroups $(T_t)_{t \geq 0}$ such that

- $T_t(f) \geq 0$ if $f \geq 0$ (positivity),
- $T_t(1) = 1$ for any $t \geq 0$ (unital, mass conservation)

are bounded analytic on $L^p(\Omega)$.

K-convex Banach spaces

A Banach space X contains ℓ_n^1's uniformly

K-convex Banach spaces

A Banach space X contains ℓ_n^1's uniformly if there exists $\lambda \geq 1$ such that for any integer $n \geq 1$ there exist

K-convex Banach spaces

A Banach space X contains ℓ_n^1's uniformly if there exists $\lambda \geq 1$ such that for any integer $n \geq 1$ there exist a n-dimensional subspace X_n of X and

K-convex Banach spaces

A Banach space X contains ℓ_n^1's uniformly if there exists $\lambda \geq 1$ such that for any integer $n \geq 1$ there exist a n-dimensional subspace X_n of X and an isomorphism $u_n \colon \ell_n^1 \to X_n$ such that $\|u_n\| \|u_n^{-1}\| < \lambda$.

K-convex Banach spaces

A Banach space X contains ℓ_n^1's uniformly if there exists $\lambda \geq 1$ such that for any integer $n \geq 1$ there exist a n-dimensional subspace X_n of X and an isomorphism $u_n \colon \ell_n^1 \to X_n$ such that $\|u_n\| \|u_n^{-1}\| < \lambda$.

A Banach space X is K-convex

K-convex Banach spaces

A Banach space X contains ℓ^1_n's uniformly if there exists $\lambda \geq 1$ such that for any integer $n \geq 1$ there exist a n-dimensional subspace X_n of X and an isomorphism $u_n \colon \ell^1_n \to X_n$ such that $\|u_n\| \|u_n^{-1}\| < \lambda$.

A Banach space X is K-convex if X does not contains ℓ^1_n's uniformly.

K-convex Banach spaces

A Banach space X contains ℓ^1_n's uniformly if there exists $\lambda \geq 1$ such that for any integer $n \geq 1$ there exist a n-dimensional subspace X_n of X and an isomorphism $u_n \colon \ell^1_n \to X_n$ such that $\|u_n\| \|u_n^{-1}\| < \lambda$.

A Banach space X is K-convex if X does not contains ℓ^1_n's uniformly.

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of independant random variables with $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$ on some probability space Ω.

K-convex Banach spaces

A Banach space X contains ℓ_n^1's uniformly if there exists $\lambda \geq 1$ such that for any integer $n \geq 1$ there exist a n-dimensional subspace X_n of X and an isomorphism $u_n \colon \ell_n^1 \to X_n$ such that $\|u_n\| \|u_n^{-1}\| < \lambda$.

A Banach space X is K-convex if X does not contains ℓ_n^1's uniformly.

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of independant random variables with $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$ on some probability space Ω.

Theorem (G. Pisier, Annals 1982)

A Banach space X is K-convex iff

K-convex Banach spaces

A Banach space X contains ℓ^1_n's uniformly if there exists $\lambda \geq 1$ such that for any integer $n \geq 1$ there exist a n-dimensional subspace X_n of X and an isomorphism $u_n \colon \ell^1_n \to X_n$ such that $\|u_n\| \|u_n^{-1}\| < \lambda$.

A Banach space X is K-convex if X does not contains ℓ^1_n's uniformly.

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of independant random variables with $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$ on some probability space Ω.

Theorem (G. Pisier, Annals 1982)

A Banach space X is K-convex iff the tensorisation $P \otimes Id_X$ of the Rademacher projection

K-convex Banach spaces

A Banach space X contains ℓ_n^1's uniformly if there exists $\lambda \geq 1$ such that for any integer $n \geq 1$ there exist a n-dimensional subspace X_n of X and an isomorphism $u_n \colon \ell_n^1 \to X_n$ such that $\|u_n\| \|u_n^{-1}\| < \lambda$.

A Banach space X is K-convex if X does not contains ℓ_n^1's uniformly.

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of independant random variables with $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$ on some probability space Ω.

Theorem (G. Pisier, Annals 1982)

A Banach space X is K-convex iff the tensorisation $P \otimes Id_X$ of the Rademacher projection

$$\begin{array}{rccc} P\colon & L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ & f & \longmapsto & \displaystyle\sum_{k=1}^{\infty} \Big(\int_\Omega f \varepsilon_k \Big) \varepsilon_k \end{array}$$

induces a bounded operator on the Bochner space $L^2(\Omega, X)$.

Preservation of the analyticity

The heart of his proof is the following result :

Preservation of the analyticity

The heart of his proof is the following result :

Theorem (G. Pisier, Annals 1982)

- *Let X be a K-convex Banach space.*

Preservation of the analyticity

The heart of his proof is the following result :

Theorem (G. Pisier, Annals 1982)

- *Let X be a K-convex Banach space.*
- *Let G be an abelian locally compact group.*

Preservation of the analyticity

The heart of his proof is the following result :

Theorem (G. Pisier, Annals 1982)

- *Let X be a K-convex Banach space.*
- *Let G be an abelian locally compact group.*
- *Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup of positive unital Fourier multipliers on $L^\infty(G)$.*

Preservation of the analyticity

The heart of his proof is the following result :

Theorem (G. Pisier, Annals 1982)

- *Let X be a K-convex Banach space.*
- *Let G be an abelian locally compact group.*
- *Let $(T_t)_{t\geq 0}$ be a symmetric diffusion semigroup of positive unital Fourier multipliers on $L^\infty(G)$.*

Then $(T_t \otimes Id_X)_{t\geq 0}$ induces a bounded analytic semigroup on the Bochner space $L^p(G, X)$ for any $1 < p < \infty$.

Preservation of the analyticity

The heart of his proof is the following result :

Theorem (G. Pisier, Annals 1982)

- *Let X be a K-convex Banach space.*
- *Let G be an abelian locally compact group.*
- *Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup of positive unital Fourier multipliers on $L^\infty(G)$.*

Then $(T_t \otimes Id_X)_{t \geq 0}$ induces a bounded analytic semigroup on the Bochner space $L^p(G, X)$ for any $1 < p < \infty$.

The property of analyticity is preserved by the tensorisation of the identity Id_X of a K-convex Banach space X.

Preservation of the analyticity

The heart of his proof is the following result :

Theorem (G. Pisier, Annals 1982)

- *Let X be a K-convex Banach space.*
- *Let G be an abelian locally compact group.*
- *Let $(T_t)_{t\geq 0}$ be a symmetric diffusion semigroup of positive unital Fourier multipliers on $L^\infty(G)$.*

Then $(T_t \otimes Id_X)_{t\geq 0}$ induces a bounded analytic semigroup on the Bochner space $L^p(G, X)$ for any $1 < p < \infty$.

The property of analyticity is preserved by the tensorisation of the identity Id_X of a K-convex Banach space X.

When we replace L^p-spaces by Bochner spaces, the semigroup remains bounded analytic.

Hence, the next question is natural :

Conjecture of Pisier

Hence, the next question is natural :

- Can we remove the assumptions “positive unital” and “of Fourier multipliers”?

Hence, the next question is natural :

- Can we remove the assumptions "positive unital" and "of Fourier multipliers"?

Conjecture (G. Pisier, 1982)

Let X be a K-convex Banach space.

Conjecture of Pisier

Hence, the next question is natural :

- Can we remove the assumptions “positive unital” and “of Fourier multipliers”?

Conjecture (G. Pisier, 1982)

Let X be a K-convex Banach space.
Let Ω be a measure space.

Conjecture of Pisier

Hence, the next question is natural :

- Can we remove the assumptions "positive unital" and "of Fourier multipliers"?

Conjecture (G. Pisier, 1982)

Let X be a K-convex Banach space.
Let Ω be a measure space.
Let $(T_t)_{t\geq 0}$ be a symmetric diffusion semigroup on $L^\infty(\Omega)$.

Hence, the next question is natural :

- Can we remove the assumptions "positive unital" and "of Fourier multipliers"?

Conjecture (G. Pisier, 1982)

Let X be a K-convex Banach space.
Let Ω be a measure space.
Let $(T_t)_{t\geq 0}$ be a symmetric diffusion semigroup on $L^\infty(\Omega)$.
Then $(T_t \otimes Id_X)_{t\geq 0}$ induces a bounded analytic semigroup on $L^p(\Omega, X)$ for any $1 < p < \infty$.

Hence, the next question is natural :

- Can we remove the assumptions “positive unital” and “of Fourier multipliers”?

Conjecture (G. Pisier, 1982)

Let X be a K-convex Banach space.
Let Ω be a measure space.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup on $L^\infty(\Omega)$.
Then $(T_t \otimes Id_X)_{t \geq 0}$ induces a bounded analytic semigroup on $L^p(\Omega, X)$ for any $1 < p < \infty$.

The scalar case $X = \mathbb{C}$ is the result of Stein.

Noncommutative mathematics

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Dictionary :

commutative	noncommutative

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Dictionary :

commutative		noncommutative
function	$\longleftrightarrow$	operator

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Dictionary :

commutative		noncommutative
function	$\longleftrightarrow$	operator
$L^\infty(\Omega)$	$\longleftrightarrow$	M von Neumann algebra

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Dictionary :

commutative		noncommutative
function	$\longleftrightarrow$	operator
$L^\infty(\Omega)$	$\longleftrightarrow$	M von Neumann algebra
$L^p(\Omega)$	$\longleftrightarrow$	$L^p(M)$

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Dictionary :

commutative		noncommutative
function	$\longleftrightarrow$	operator
$L^\infty(\Omega)$	$\longleftrightarrow$	M von Neumann algebra
$L^p(\Omega)$	$\longleftrightarrow$	$L^p(M)$
ℓ^p	$\longleftrightarrow$	S^p

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Dictionary :

commutative		noncommutative
function	$\longleftrightarrow$	operator
$L^\infty(\Omega)$	$\longleftrightarrow$	M von Neumann algebra
$L^p(\Omega)$	$\longleftrightarrow$	$L^p(M)$
ℓ^p	$\longleftrightarrow$	S^p
$\left(\int_\Omega \lvert f\rvert^p d\mu\right)^{\frac{1}{p}}$	$\longleftrightarrow$	$\lVert x\rVert_{S^p} = \left(\mathrm{Tr}\,\lvert x\rvert^p\right)^{\frac{1}{p}}$ où $\lvert x\rvert = (x^*x)^{\frac{1}{2}}$

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Dictionary :

commutative		noncommutative
function	$\longleftrightarrow$	operator
$L^\infty(\Omega)$	$\longleftrightarrow$	M von Neumann algebra
$L^p(\Omega)$	$\longleftrightarrow$	$L^p(M)$
ℓ^p	$\longleftrightarrow$	S^p
$\left(\int_\Omega \|f\|^p d\mu\right)^{\frac{1}{p}}$	$\longleftrightarrow$	$\\|x\\|_{S^p} = \left(\mathrm{Tr}\, \|x\|^p\right)^{\frac{1}{p}}$ où $\|x\| = (x^*x)^{\frac{1}{2}}$
Banach space X	$\longleftrightarrow$	operator space E :

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Dictionary :

commutative		noncommutative
function	$\longleftrightarrow$	operator
$L^\infty(\Omega)$	$\longleftrightarrow$	M von Neumann algebra
$L^p(\Omega)$	$\longleftrightarrow$	$L^p(M)$
ℓ^p	$\longleftrightarrow$	S^p
$\left(\int_\Omega \lvert f\rvert^p d\mu\right)^{\frac{1}{p}}$	$\longleftrightarrow$	$\lVert x\rVert_{S^p} = \left(\mathrm{Tr}\,\lvert x\rvert^p\right)^{\frac{1}{p}}$ où $\lvert x\rvert = (x^*x)^{\frac{1}{2}}$
Banach space X	$\longleftrightarrow$	operator space E : Banach space E equipped with an embedding $E \hookrightarrow B(H)$

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Dictionary :

commutative		noncommutative
function	$\longleftrightarrow$	operator
$L^\infty(\Omega)$	$\longleftrightarrow$	M von Neumann algebra
$L^p(\Omega)$	$\longleftrightarrow$	$L^p(M)$
ℓ^p	$\longleftrightarrow$	S^p
$\left(\int_\Omega \lvert f\rvert^p d\mu\right)^{\frac{1}{p}}$	$\longleftrightarrow$	$\lVert x\rVert_{S^p} = \left(\mathrm{Tr}\, \lvert x\rvert^p\right)^{\frac{1}{p}}$ où $\lvert x\rvert = (x^*x)^{\frac{1}{2}}$
Banach space X	$\longleftrightarrow$	operator space E : Banach space E equipped with an embedding $E \hookrightarrow B(H)$
$L^p(\Omega, X)$	$\longleftrightarrow$	$L^p(M, E)$

Noncommutative mathematics

To provide a partial answer to this question, we use noncommutative methods.

Dictionary :

commutative		noncommutative
function	$\longleftrightarrow$	operator
$L^\infty(\Omega)$	$\longleftrightarrow$	M von Neumann algebra
$L^p(\Omega)$	$\longleftrightarrow$	$L^p(M)$
ℓ^p	$\longleftrightarrow$	S^p
$\left(\int_\Omega \|f\|^p d\mu\right)^{\frac{1}{p}}$	$\longleftrightarrow$	$\\|x\\|_{S^p} = \left(\mathrm{Tr}\, \|x\|^p\right)^{\frac{1}{p}}$ où $\|x\| = (x^*x)^{\frac{1}{2}}$
Banach space X	$\longleftrightarrow$	operator space E : Banach space E equipped with an embedding $E \hookrightarrow B(H)$
$L^p(\Omega, X)$	$\longleftrightarrow$	$L^p(M, E)$
X K-convex	$\longleftrightarrow$	E OK-convex $\Longleftrightarrow$ $S^p(E)$ is K-convex.

A partial answer

Theorem (C. A., 2014)

Let G be an abelian compact group.

A partial answer

Theorem (C. A., 2014)

Let G be an abelian compact group.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup of Fourier multipliers on $L^\infty(G)$.

A partial answer

Theorem (C. A., 2014)

Let G be an abelian compact group.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup of Fourier multipliers on $L^\infty(G)$.

Let X be a K-convex Banach space

A partial answer

Theorem (C. A., 2014)

Let G be an abelian compact group.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup of Fourier multipliers on $L^\infty(G)$.

Let X be a K-convex Banach space isomorphic to a Banach space E which admits an OK-convex operator space structure.

A partial answer

Theorem (C. A., 2014)

Let G be an abelian compact group.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup of Fourier multipliers on $L^\infty(G)$.

Let X be a K-convex Banach space isomorphic to a Banach space E which admits an OK-convex operator space structure.

Then $(T_t \otimes Id_X)_{t \geq 0}$ induces a bounded analytic semigroup on the Bochner space $L^p(G, X)$ for any $1 < p < \infty$.

A partial answer

Theorem (C. A., 2014)

Let G be an abelian compact group.
Let $(T_t)_{t\geq 0}$ be a symmetric diffusion semigroup of Fourier multipliers on $L^\infty(G)$.

Let X be a K-convex Banach space isomorphic to a Banach space E which admits an OK-convex operator space structure.

Then $(T_t \otimes Id_X)_{t\geq 0}$ induces a bounded analytic semigroup on the Bochner space $L^p(G, X)$ for any $1 < p < \infty$.

- Let X be a K-convex Banach space.

A partial answer

Theorem (C. A., 2014)

Let G be an abelian compact group.
Let $(T_t)_{t\geq 0}$ be a symmetric diffusion semigroup of Fourier multipliers on $L^\infty(G)$.

Let X be a K-convex Banach space isomorphic to a Banach space E which admits an OK-convex operator space structure.

Then $(T_t \otimes Id_X)_{t\geq 0}$ induces a bounded analytic semigroup on the Bochner space $L^p(G, X)$ for any $1 < p < \infty$.

- Let X be a K-convex Banach space.
  Does there exists a Banach space E wich admits an OK-convex operator space structure and which is isomorphic to X ?

A partial answer

Theorem (C. A., 2014)

Let G be an abelian compact group.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup of Fourier multipliers on $L^\infty(G)$.

Let X be a K-convex Banach space isomorphic to a Banach space E which admits an OK-convex operator space structure.

Then $(T_t \otimes Id_X)_{t \geq 0}$ induces a bounded analytic semigroup on the Bochner space $L^p(G, X)$ for any $1 < p < \infty$.

- Let X be a K-convex Banach space.
  Does there exists a Banach space E wich admits an OK-convex operator space structure and which is isomorphic to X ?
  i.e. such that the Banach space $S^p(E)$ is K-convex ?

A partial answer

Theorem (C. A., 2014)

Let G be an abelian compact group.
Let $(T_t)_{t \geq 0}$ be a symmetric diffusion semigroup of Fourier multipliers on $L^\infty(G)$.

Let X be a K-convex Banach space isomorphic to a Banach space E which admits an OK-convex operator space structure.

Then $(T_t \otimes Id_X)_{t \geq 0}$ induces a bounded analytic semigroup on the Bochner space $L^p(G, X)$ for any $1 < p < \infty$.

- Let X be a K-convex Banach space.
  Does there exists a Banach space E wich admits an OK-convex operator space structure and which is isomorphic to X ?
  i.e. such that the Banach space $S^p(E)$ is K-convex ?
- The latter problem is part of a more general framework :
  a similar question exists for other Banach spaces properties (UMD, cotype...).

Schur multipliers

Schur multipliers

An operator in $B(\ell^2)$ can be identified with its corresponding matrix relatively to the canonical basis of ℓ^2.

An operator in $B(\ell^2)$ can be identified with its corresponding matrix relatively to the canonical basis of ℓ^2.

A Schur multiplier defined by a matrix A is a map

$$\begin{array}{rccc} M_A: & B(\ell^2) & \longrightarrow & B(\ell^2) \\ & [x_{ij}] & \longmapsto & [a_{ij}x_{ij}]. \end{array}$$

Schur multipliers

An operator in $B(\ell^2)$ can be identified with its corresponding matrix relatively to the canonical basis of ℓ^2.

A Schur multiplier defined by a matrix A is a map

$$\begin{array}{cccc} M_A \colon & B(\ell^2) & \longrightarrow & B(\ell^2) \\ & [x_{ij}] & \longmapsto & [a_{ij}x_{ij}]. \end{array}$$

Recall that

$$S^p = \left\{ x \in B(\ell^2) \colon \|x\|_{S^p} = \left(\mathrm{Tr}\,|x|^p\right)^{\frac{1}{p}} < \infty \right\}$$

where $|x| = (x^* x)^{\frac{1}{2}}$.

Semigroups of Schur multipliers

Theorem (C. A., 2014)

Let $(T_t)_{t \geq 0}$ be a w^*-semigroup of contractive Schur multipliers on $B(\ell^2)$ which are selfadjoint on S^2.

Theorem (C. A., 2014)

Let $(T_t)_{t \geq 0}$ be a w^*-semigroup of contractive Schur multipliers on $B(\ell^2)$ which are selfadjoint on S^2.

Let E be an OK-convex operator space.

Semigroups of Schur multipliers

Theorem (C. A., 2014)

Let $(T_t)_{t\geq 0}$ be a w^*-semigroup of contractive Schur multipliers on $B(\ell^2)$ which are selfadjoint on S^2.

Let E be an OK-convex operator space.

Then $(T_t \otimes Id_E)_{t\geq 0}$ induces a bounded analytic semigroup on $S^p(E)$ for any $1 < p < \infty$.

Semigroups of Schur multipliers

Theorem (C. A., 2014)

Let $(T_t)_{t\geq 0}$ be a w^*-semigroup of contractive Schur multipliers on $B(\ell^2)$ which are selfadjoint on S^2.

Let E be an OK-convex operator space.

Then $(T_t \otimes Id_E)_{t\geq 0}$ induces a bounded analytic semigroup on $S^p(E)$ for any $1 < p < \infty$.

Theorem (C. A., 2014)

The w^*-semigroups of contractive Schur multipliers on $B(\ell^2)$ which are selfadjoint on S^2

Theorem (C. A., 2014)

Let $(T_t)_{t \geq 0}$ be a w^*-semigroup of contractive Schur multipliers on $B(\ell^2)$ which are selfadjoint on S^2.

Let E be an OK-convex operator space.

Then $(T_t \otimes Id_E)_{t \geq 0}$ induces a bounded analytic semigroup on $S^p(E)$ for any $1 < p < \infty$.

Theorem (C. A., 2014)

The w^*-semigroups of contractive Schur multipliers on $B(\ell^2)$ which are selfadjoint on S^2 are precisely the semigroups

$$T_t = \left[\mathrm{e}^{-t\|\alpha_i - \beta_j\|_H^2}\right]_{i,j \geq 1},$$

where α_i, β_j are elements of a real Hilbert space H.

Beurling Theorem

Theorem (A. Beurling, 1970)

Let X be a Banach space.

Beurling Theorem

Theorem (A. Beurling, 1970)

Let X be a Banach space.
Let $(T_t)_{t\geq 0}$ be a C_0-semigroup of contractions on X.

Beurling Theorem

Theorem (A. Beurling, 1970)

Let X be a Banach space.
Let $(T_t)_{t \geq 0}$ be a C_0-semigroup of contractions on X.
Suppose that there exists an integer $n \geq 1$ such that

$$\limsup_{t \to 0} \left\| (Id_X - T_t)^n \right\|_{X \to X} < 2^n.$$

Beurling Theorem

Theorem (A. Beurling, 1970)

Let X be a Banach space.
Let $(T_t)_{t\geq 0}$ be a C_0-semigroup of contractions on X.
Suppose that there exists an integer $n \geq 1$ such that

$$\limsup_{t\to 0} \left\|(Id_X - T_t)^n\right\|_{X\to X} < 2^n.$$

Then the semigroup $(T_t)_{t\geq 0}$ is bounded analytic.

Beurling Theorem

Theorem (A. Beurling, 1970)

Let X be a Banach space.
Let $(T_t)_{t \geq 0}$ be a C_0-semigroup of contractions on X.
Suppose that there exists an integer $n \geq 1$ such that

$$\limsup_{t \to 0} \left\| (Id_X - T_t)^n \right\|_{X \to X} < 2^n.$$

Then the semigroup $(T_t)_{t \geq 0}$ is bounded analytic.

The regularity of a semigroup is, roughly spoken, induced by the quality of the approximation of the identity operator as the parameter goes to zero.

K-convexity and projections

Lemme (G. Pisier, 1982)

Let X be a K-convex Banach space.

Lemme (G. Pisier, 1982)

Let X be a K-convex Banach space.

Then there exists $0 < \rho < 2$ and an integer n

Lemme (G. Pisier, 1982)

Let X be a K-convex Banach space.

Then there exists $0 < \rho < 2$ and an integer n such that any commuting norm-one projections $P_1, \dots, P_n$ on X satisfy

Lemme (G. Pisier, 1982)

Let X be a K-convex Banach space.

Then there exists $0 < \rho < 2$ and an integer n such that any commuting norm-one projections $P_1, \ldots, P_n$ on X satisfy

$$\left\| \prod_{1 \leq k \leq n} (Id_X - P_k) \right\|_{X \to X} \leq \rho^n < 2^n.$$

Rota Dilation

Definition

Let M be a von Neumann algebra ($=$ noncommutative $L^\infty(\Omega)$ space)

Rota Dilation

Definition

Let M be a von Neumann algebra (= noncommutative $L^\infty(\Omega)$ space) equipped with a trace (= noncommutative integral) .

Rota Dilation

Definition

Let M be a von Neumann algebra (= noncommutative $L^\infty(\Omega)$ space) equipped with a trace (= noncommutative integral) .
An operator $T: M \to M$ satisfies Rota dilation property if there exist

Rota Dilation

Definition

Let M be a von Neumann algebra (= noncommutative $L^\infty(\Omega)$ space) equipped with a trace (= noncommutative integral) .
An operator $T: M \to M$ satisfies Rota dilation property if there exist

$$\begin{tikzcd} N \arrow[r, "\mathbb{E}_k"] & N \arrow[d, "\mathbb{E}"] \\ M \arrow[u, "J"] \arrow[r, "T^k"] & M \end{tikzcd}$$

Rota Dilation

Definition

Let M be a von Neumann algebra ($=$ noncommutative $L^\infty(\Omega)$ space) equipped with a trace ($=$ noncommutative integral) .
An operator $T \colon M \to M$ satisfies Rota dilation property if there exist

$$\begin{tikzcd} N \arrow[r, "\mathbb{E}_k"] & N \arrow[d, "\mathbb{E}"] \\ M \arrow[u, "J"] \arrow[r, "T^k"] & M \end{tikzcd}$$

- *where $k \in \mathbb{N}$ and N is a vN algebra equipped with a trace,*

Rota Dilation

Definition

Let M be a von Neumann algebra ($=$ noncommutative $L^\infty(\Omega)$ space) equipped with a trace ($=$ noncommutative integral) .
An operator $T\colon M \to M$ satisfies Rota dilation property if there exist

$$\begin{tikzcd} N \arrow[r, "\mathbb{E}_k"] & N \arrow[d, "\mathbb{E}"] \\ M \arrow[u, "J"] \arrow[r, "T^k"] & M \end{tikzcd}$$

- *where $k \in \mathbb{N}$ and N is a vN algebra equipped with a trace,*
- *$J\colon M \to N$ is a normal unital faithful $*$-representation which preserves traces,*

Rota Dilation

Definition

Let M be a von Neumann algebra ($=$ noncommutative $L^\infty(\Omega)$ space) equipped with a trace ($=$ noncommutative integral) .
An operator $T\colon M \to M$ satisfies Rota dilation property if there exist

$$\begin{array}{ccc} N & \xrightarrow{\mathbb{E}_k} & N \\ {\scriptstyle J}\uparrow & & \downarrow{\scriptstyle \mathbb{E}} \\ M & \xrightarrow{T^k} & M \end{array}$$

- *where $k \in \mathbb{N}$ and N is a vN algebra equipped with a trace,*
- *$J\colon M \to N$ is a normal unital faithful $*$-representation which preserves traces,*
- *$\mathbb{E}\colon N \to M$ is the conditional expectation associated with J,*

Rota Dilation

Definition

Let M be a von Neumann algebra ($=$ noncommutative $L^\infty(\Omega)$ space) equipped with a trace ($=$ noncommutative integral) .
An operator $T\colon M \to M$ satisfies Rota dilation property if there exist

$$\begin{tikzcd} N \arrow[r, "\mathbb{E}_k"] & N \arrow[d, "\mathbb{E}"] \\ M \arrow[u, "J"] \arrow[r, "T^k"] & M \end{tikzcd}$$

- *where $k \in \mathbb{N}$ and N is a vN algebra equipped with a trace,*
- *$J\colon M \to N$ is a normal unital faithful $*$-representation which preserves traces,*
- *$\mathbb{E}\colon N \to M$ is the conditional expectation associated with J,*
- *$N_1 \supset N_2 \supset \cdots$ is a decreasing sequence of subalgebras of N associated to conditional expectations $\mathbb{E}_k$.*

Proof for the semigroups of positive Schur multipliers

There exists a dilation for each positive selfadjoint unital Schur multiplier T_t (É. Ricard, 2008)

Proof for the semigroups of positive Schur multipliers

There exists a dilation for each positive selfadjoint unital Schur multiplier T_t (É. Ricard, 2008) with $\mathbb{E}_k \colon N \to N$:

$$T_t^k = \mathbb{E}\mathbb{E}_k J, \quad \text{hence} \quad T_t = \mathbb{E}\mathbb{E}_1 J.$$

Proof for the semigroups of positive Schur multipliers

There exists a dilation for each positive selfadjoint unital Schur multiplier T_t (É. Ricard, 2008) with $\mathbb{E}_k \colon N \to N$:

$$T_t^k = \mathbb{E}\mathbb{E}_k J, \quad \text{hence} \quad T_t = \mathbb{E}\mathbb{E}_1 J.$$

We define the commuting contractive projections on $L^p(N^{\otimes n}, E)$:

Proof for the semigroups of positive Schur multipliers

There exists a dilation for each positive selfadjoint unital Schur multiplier T_t (É. Ricard, 2008) with $\mathbb{E}_k \colon N \to N$:

$$T_t^k = \mathbb{E}\mathbb{E}_k J, \quad \text{hence} \quad T_t = \mathbb{E}\mathbb{E}_1 J.$$

We define the commuting contractive projections on $L^p(N^{\otimes n}, E)$:

$$\begin{aligned}
P_1 &= \big(\mathbb{E}_1 \otimes Id \otimes \cdots \otimes Id\big) \otimes Id_E, \\
P_2 &= \big(Id \otimes \mathbb{E}_1 \otimes Id \otimes \cdots \otimes Id\big) \otimes Id_E, \\
&\ \ \ldots \\
P_n &= \big(Id \otimes \cdots \otimes Id \otimes \mathbb{E}_1\big) \otimes Id_E.
\end{aligned}$$

Proof for the semigroups of positive Schur multipliers

There exists a dilation for each positive selfadjoint unital Schur multiplier T_t (É. Ricard, 2008) with $\mathbb{E}_k \colon N \to N$:

$$T_t^k = \mathbb{E}\mathbb{E}_k J, \qquad \text{hence} \qquad T_t = \mathbb{E}\mathbb{E}_1 J.$$

We define the commuting contractive projections on $L^p(N^{\otimes n}, E)$:

$$\begin{aligned} P_1 &= (\mathbb{E}_1 \otimes Id \otimes \cdots \otimes Id) \otimes Id_E, \\ P_2 &= (Id \otimes \mathbb{E}_1 \otimes Id \otimes \cdots \otimes Id) \otimes Id_E, \\ &\ldots \\ P_n &= (Id \otimes \cdots \otimes Id \otimes \mathbb{E}_1) \otimes Id_E. \end{aligned}$$

We have $$(Id - T_t)^{\otimes n} \otimes Id_E = \mathbb{E}^{\otimes n}\left(\prod_{1 \leq k \leq n} (Id - P_k)\right) J^{\otimes n}.$$

Proof for the semigroups of positive Schur multipliers

There exists a dilation for each positive selfadjoint unital Schur multiplier T_t (É. Ricard, 2008) with $\mathbb{E}_k \colon N \to N$:

$$T_t^k = \mathbb{E}\mathbb{E}_k J, \qquad \text{hence} \qquad T_t = \mathbb{E}\mathbb{E}_1 J.$$

We define the commuting contractive projections on $L^p(N^{\otimes n}, E)$:

$$\begin{aligned}
P_1 &= \big(\mathbb{E}_1 \otimes Id \otimes \cdots \otimes Id\big) \otimes Id_E, \\
P_2 &= \big(Id \otimes \mathbb{E}_1 \otimes Id \otimes \cdots \otimes Id\big) \otimes Id_E, \\
&\ \ \ldots \\
P_n &= \big(Id \otimes \cdots \otimes Id \otimes \mathbb{E}_1\big) \otimes Id_E.
\end{aligned}$$

We have $$\big(Id - T_t\big)^{\otimes n} \otimes Id_E = \mathbb{E}^{\otimes n}\Bigg(\prod_{1 \leq k \leq n} \big(Id - P_k\big)\Bigg) J^{\otimes n}.$$

The norm of $\big(Id - T_t\big)^n \otimes Id_E$ is bounded by the norm of $\big(Id - T_t\big)^{\otimes n} \otimes Id_E$.

Absorption Lemma

The latter inequality is a consequence of :

Absorption Lemma

The latter inequality is a consequence of :

Lemma (C.A., 2014)

Let E be an operator space.

Absorption Lemma

The latter inequality is a consequence of :

Lemma (C.A., 2014)

Let E be an operator space.
Let $n \geq 1$ be an integer.

Absorption Lemma

The latter inequality is a consequence of :

Lemma (C.A., 2014)

Let E be an operator space.
Let $n \geq 1$ be an integer.
For any matrix $x = [x_{ij}]_{i,j\geq 1}$ with entries in E finitely supported, we have

Absorption Lemma

The latter inequality is a consequence of :

Lemma (C.A., 2014)

Let E be an operator space.
Let $n \geq 1$ be an integer.
For any matrix $x = [x_{ij}]_{i,j \geq 1}$ with entries in E finitely supported, we have

$$\left\| \sum_{i,j} e_{ij} \otimes e_{ij} \otimes \cdots \otimes e_{ij} \otimes x_{ij} \right\|_{S^p(B(\ell^2)^{\otimes n}, E)} = \left\| \sum_{i,j} e_{ij} \otimes x_{ij} \right\|_{S^p(E)}.$$

It was a pleasure to present this work to you !