Geometry and the Kato Square Root Problem

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Functional Caculus and Harmonic Analysis of Semigroups

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Let $A \in \mathrm{L}^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$ and $a \in \mathrm{L}^\infty(\mathbb{R}^n)$. Suppose that there exists $\kappa_1, \kappa_2 > 0$ such that for all $u \in \mathrm{W}^{1,2}(\mathbb{R}^n)$,

$$\mathrm{Re}\, a(x) \geq \kappa_1 \quad \text{and} \quad \mathrm{Re}\, \langle A \nabla u, \nabla u \rangle \geq \kappa_2 \|u\|^2.$$

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$$\begin{aligned} \mathcal{D}(\sqrt{-a \operatorname{div} A\nabla}) &= \mathrm{W}^{1,2}(\mathbb{R}^n) \\ \|\sqrt{-a \operatorname{div} A\nabla} u\| &\simeq \|\nabla u\|. \end{aligned} \tag{K1}$$

This was answered in the positive in 2002 by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh and Phillipe Tchamitchian in [AHLMcT].

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Consider the following second order differential operator $\mathrm{L}_A : \mathcal{D}(\mathrm{L}_A) \subset \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ defined by

$$\mathrm{L}_A u = aS^*ASu = -a\,\mathrm{div}(A_{11}\nabla u) - a\,\mathrm{div}(A_{10}u) + aA_{01}\nabla u + aA_{00}u.$$

The main theorem on manifolds

Theorem (B.-Mc, 2012)

Let $\mathcal{M}$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$*. Suppose the following ellipticity condition holds: there exists* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$$
$$\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2 \|u\|^2_{\mathrm{W}^{1,2}}$$

for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{M})$*. Then,*
$\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{M})$ *and*
$\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{M})$.

Lipschitz estimates

Since we allow the coefficients a and A to be *complex*, we obtain the following stability result as a consequence:

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$$\|\sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\|u\|_{\mathrm{W}^{1,2}}$$

holds for all $u \in \mathrm{W}^{1,2}(\mathcal{M})$. The implicit constant depends in particular on A, a and η_i.

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These results are the contents of the preprint [BMc].

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For an invertible $A \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M})))$, we consider perturbing D to obtain the operator $\mathrm{D}_A = \mathrm{d} + A^{-1}\mathrm{d}^* A$.

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$$\mathrm{R}\,\omega = \mathrm{Rm}_{ijkl}\,\theta^i \wedge (\theta^j \llcorner (\theta^k \wedge (\theta^l \llcorner \omega)))$$

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This can be seen as an extension of Ricci curvature for forms, since $\mathrm{g}(\mathrm{R}\,\omega, \eta) = \mathrm{Ric}(\omega^\flat, \eta^\flat)$ whenever $\omega, \eta \in \boldsymbol{\Omega}_x^1(\mathcal{M})$ and where $\flat : \mathrm{T}^*\mathcal{M} \to \mathrm{T}\mathcal{M}$ is the flat isomorphism through the metric g.

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The Weitzenböck formula then asserts that $\mathrm{D}^2 = \mathrm{tr}_{12}\,\nabla^2 + \mathrm{R}$.

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Let $\mathcal{M}$ be a smooth, complete Riemannian manifold and let $\beta \in \mathbb{C} \setminus \{0\}$. Suppose there exist $\eta, \kappa > 0$ such that $|\mathrm{Ric}| \leq \eta$ and $\mathrm{inj}(\mathcal{M}) \geq \kappa$. Furthermore, suppose there exists $\zeta \in \mathbb{R}$ satisfying

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for $u \in \mathbf{\Omega}_x(\mathcal{M})$, and $A \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M})))$ and $\kappa_1 > 0$ satisfying

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This result is included in my thesis, [B-Thesis].

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- Poincaré inequality - on both functions and vector fields.
- Control of ∇^2 in terms of Δ.

Rough metrics

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Definition (Rough metric)

Let g be a $(2,0)$ symmetric tensor field with measurable coefficients and that for each $x \in \mathcal{M}$, there is some chart (U, ψ) near x and a constant $C \geq 1$ such that

$$C^{-1} \left|u\right|_{\psi^* \delta(y)} \leq \left|u\right|_{\mathrm{g}(y)} \leq C \left|u\right|_{\psi^* \delta(y)},$$

for almost-every $y \in U$ and where δ is the Euclidean metric in $\psi(U)$. Then we say that g is a rough metric, and such a chart (U, ψ) is said to satisfy the *local comparability condition*.

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for almost-every $y \in U$ and where δ is the Euclidean metric in $\psi(U)$. Then we say that g is a rough metric, and such a chart (U, ψ) is said to satisfy the *local comparability condition*.

Rough metrics may not induce a well defined length structure, but they induce a Borel volume measure that is finite on compact sets.

Metric perturbations

We say that two rough metrics g and $\tilde{\mathrm{g}}$ are C-close

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Suppose that g is a rough metric that is C-close to a continuous metric $\tilde{\mathrm{g}}$ and that the Kato square root problem can be solved on $(\mathcal{M}, \tilde{\mathrm{g}})$. Then, the Kato square root problem can be solved on g.

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In particular, the Kato square root problem can be solved on every smooth compact manifold equipped with a rough metric.

This has consequences to the spectral theory of the associated Laplacian on such a geometry.

Lipschitz transformations

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Such a metric g has measurable coefficients and is, in fact, a rough metric.

Theorem (B., 2014)

The Kato square root problem for functions is invariant under Lipschitz transformations.

Lower bounds on injectivity radius

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Theorem (B., 2014)

For any $C > 1$, there exists a smooth metric g on $\mathbb{R}^2$ such that $\mathrm{inj}(\mathcal{M}, \mathrm{g}) = 0$ and it is C-close to the standard Euclidean metric. Thus, the Kato square root problem can be solved on $(\mathcal{M}, \mathrm{g})$.

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These results can be found in [B].

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