# From resolvent estimates to rates of decay

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Functional Calculus and Harmonic Analysis of Semigroups
Besançon, 1 October 2014

# A damped wave equation

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \Delta u + 2a(x)\frac{\partial u}{\partial t} &= 0 \qquad (t > 0, x \in \Omega) \\
u(x,t) &= 0 \qquad (t > 0, x \in \partial\Omega)
\end{aligned}$$

$$u(\cdot, 0) = u_0 \in H_0^1(\Omega), \qquad \frac{\partial u}{\partial t}(\cdot, 0) = u_1 \in L^2(\Omega).$$

Here, $\Omega$ is a compact Riemannian manifold (with boundary), and $a : \Omega \to [0, \infty)$ (continuous).

Energy

$$E(u,t) = \frac{1}{2}\int_\Omega \left( |\nabla u|^2 + \left|\frac{\partial u}{\partial t}\right|^2 \right) \, dx,$$

decreasing in $t$.

# Energy decay

Except in degenerate cases,

1. the energy $E(u,t) \to 0$ as $t \to \infty$;
2. if the domain of damping $\{x : a(x) > 0\}$ satisfies the geometric optics condition then the decay occurs at an exponential rate (Bardos-Lebeau-Rauch, 1992);
3. in other cases, the decay occurs at a polynomial rate or a logarithmic rate, uniformly for smooth initial data.

# Operator formulation

Reformulate the damped wave equation:

$$\begin{aligned} X &= H_0^1(\Omega) \times L^2(\Omega), \\ A &= \begin{pmatrix} 0 & 1 \\ \Delta & -2a(x) \end{pmatrix}, \\ D(A) &= (H^2 \cap H_0^1) \times H_0^1. \end{aligned}$$

$$U(t) = \begin{pmatrix} u(t) \\ \frac{\partial u}{\partial t} \end{pmatrix} \in X, \qquad E(u,t) = \tfrac{1}{2} \| U(t) \|^2_{H^1 \times L^2},$$

$$U'(t) = AU(t), \qquad U(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Lebeau (1996) established that

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} : -2\|a\|_\infty \leq \operatorname{Re}\lambda < 0\},$$

and $\left\|(is - A)^{-1}\right\|$ grows (at most) polynomially as $|s| \to \infty$.

The damped wave equation is well-posed, so $A$ generates a $C_0$-semigroup of contractions $\{T(t) : t \geq 0\}$ on $X$.

Decay of $E(u, t)$ for initial data $(u_0, u_1) \in D(A)$ corresponds to decay of $\|T(t)(\lambda - A)^{-1}\|$ for any $\lambda \in \rho(A)$. One establishes such decay by some form of complex inversion of Laplace transforms.

# An abstract theorem

$X$ complex Banach space, $\{T(t) : t \geq 0\}$ $C_0$-semigroup, generator $A$

If $T$ is bounded, $\qquad \sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq 0\}$

**Theorem**

*If $T$ is bounded and $\sigma(A) \cap i\mathbb{R}$ is empty, then*

$$\lim_{t\to\infty} \|T(t)A^{-1}\| = 0.$$

There is also a converse result.

Consequence:

$$\lim_{t\to\infty} \|T(t)x\| = 0 \qquad (x \in X).$$

# Quantified version

$M : \mathbb{R}_+ \to (0, \infty)$, continuous, increasing.

$M_{\log}(s) = M(s)(\log(1 + M(s)) + \log(1 + s))$.

**Theorem ($M_{\log}$-Theorem; B-Duyckaerts 2008)**

*Suppose that $T$ is bounded and*

$$\left\|(is - A)^{-1}\right\| \le M(|s|) \quad (s \in \mathbb{R}).$$

*Then, for any $c \in (0, 1)$,*

$$\left\|T(t)A^{-1}\right\| = O\left(\frac{1}{M_{\log}^{-1}(ct)}\right) \quad (t \to \infty).$$

# Examples

1. $M(s) = C$: one obtains exponential decay of $\|T(t)A^{-1}\|$. (True more generally, Weis-Wrobel)
2. $M(s) = C\log(2+s) \implies \|T(t)A^{-1}\| = O\left(e^{-c\sqrt{t}}\right)$
3. $M(s) = C\exp(\alpha s)$: one obtains logarithmic decay (Lebeau, Burq for Hilbert space)
4. $M(s) = C(1+s)^{\alpha}$: one obtains polynomial stability:

$$\|(is - A)^{-1}\| \leq C(1+|s|)^{\alpha} \implies \|T(t)A^{-1}\| = O\left(\left(\frac{\log t}{t}\right)^{\frac{1}{\alpha}}\right)$$

slightly sharper than Liu and Rao (Hilbert space) and Batkai et al (Banach spaces)

# Necessary condition

**Theorem**

*Let* $m : (0,\infty) \to (0,\infty)$ *be decreasing with* $\lim_{t\to\infty} m(t) = 0$. *Assume that*

$$\left\| T(t)(1-A)^{-1} \right\| \le m(t) \quad (t > 0).$$

*Then* $\sigma(A) \cap i\mathbb{R}$ *is empty, and, for each* $c \in (0,1)$,

$$\left\| (is-A)^{-1} \right\| = O\left( m^{-1}(c/|s|) \right) \quad (|s| \to \infty).$$

So the apparently optimal rate of decay in Ingham-Karamata Theorem would have $M^{-1}$ instead of $M_{\log}^{-1}$.

## Theorem (Borichev-Tomilov 2010)

*Let $M(s) = C(1+s)^\alpha$. On general Banach spaces, it is not possible to improve the conclusion of the $M_{\log}$-theorem that the rate of decay is*

$$O\left(\left(\frac{\log t}{t}\right)^{\frac{1}{\alpha}}\right) \quad (t \to \infty).$$

*In the case of a bounded $C_0$-semigroup on a Hilbert space satisfying*

$$\left\|(is - A)^{-1}\right\| \leq C(1 + |s|)^\alpha \quad (s \in \mathbb{R}),$$

*one has*

$$\left\|T(t)A^{-1}\right\| = O\left(\frac{1}{t^{\frac{1}{\alpha}}}\right) \quad (t \to \infty).$$

# Regularly varying case

$M$ is *regularly varying* if $M(s) \sim \frac{s^\alpha}{\ell(s)}$ where $\ell$ is slowly varying.

Consider the case $\alpha = 1$ (purely for simplicity), $\ell$ increasing.

**Theorem (B-Chill-Tomilov, to appear, JEMS)**

*Assume that $X$ is a Hilbert space, and $\ell$ is slowly varying and increasing, and*

$$\|(is - A)^{-1}\| = O\left(\frac{|s|}{\ell(|s|)}\right) \quad (|s| \to \infty).$$

*Then*

$$\|T(t)A^{-1}\| = O\left(\frac{1}{t\ell(t)}\right) \quad (t \to \infty).$$

For many (but not all) $\ell$, this gives the optimal result $\|T(t)A^{-1}\| = O(1/M^{-1}(t))$.

### Theorem (B-Chill-Tomilov, to appear, JEMS)

*Assume that $X$ is a Hilbert space, and $\ell$ is slowly varying and decreasing, and*

$$\left\|(is - A)^{-1}\right\| = O\left(\frac{|s|}{\ell(|s|)}\right) \quad (|s| \to \infty).$$

*Then, for every $\varepsilon > 0$,*

$$\left\|T(t)A^{-1}\right\| = O\left(\frac{(\log t)^{\varepsilon}}{t\widetilde{\ell}(t)}\right) \quad (t \to \infty).$$

Here $\widetilde{\ell}$ is a slowly varying function which is sometimes, but not always, the same as $\ell$. However the optimal result would have $\varepsilon = 0$.

# Laplace transforms

$$\Omega_M := \left\{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\frac{1}{M(|\operatorname{Im} \lambda|)} \right\}$$

**Theorem (Ingham, Karamata, 1935; ..... , B-Duyckaerts 2008)**

*Let $f \in L^\infty(\mathbb{R}_+, X)$, and assume that $\widehat{f}$ extends analytically to $\Omega_M$ and the extension satisfies*

$$\|\widehat{f}(\lambda)\| \le M(|\operatorname{Im} \lambda|), \qquad \lambda \in \Omega_M.$$

*Let $c \in (0,1)$. Then there exist positive numbers $C$ and $t_0$, depending only on $\|f\|_\infty$, $M$ and $c$, such that*

$$\left\| \widehat{f}(0) - \int_0^t f(s)\, ds \right\| \le \frac{C}{M_{\log}^{-1}(ct)}, \qquad t \ge t_0.$$

The semigroup theorem follows by taking $f(t) = T(t)x$.

# $L^p$-version

**Theorem (B-Borichev-Tomilov)**

*Let $f \in L^p(\mathbb{R}_+, X)$, where $1 \le p \le \infty$, and assume that $\widehat{f}$ extends analytically to $\Omega_M$ and satisfies*

$$\|\widehat{f}(\lambda)\| \le M(|\operatorname{Im}\lambda|), \qquad \lambda \in \Omega_M.$$

*Then there exists $c > 0$, depending only on $p$, such that the function*

$$t \mapsto M_{\log}^{-1}(ct)\left(\widehat{f}(0) - \int_0^t f(s)\,ds\right)$$

*belongs to $L^p(\mathbb{R}_+, X)$.*

The shape of $\Omega_M$ is much more important than the bound on $\widehat{f}$.

## Theorem

*Let $(T(t)) : t \geq 0$ be a $C_0$-semigroup on a Banach space $X$, with generator $A$, and let $\omega \in \rho(A)$, $S \in \mathcal{B}(X)$. If $x \in X$ is such that*

(i) *$ST(\cdot)x \in L^p(\mathbb{R}_+, X)$ for some $p \geq 1$, and*

(ii) *$SR(\cdot, A)x$ extends to an analytic function $G$ on an open set $\Omega$ containing $\overline{\mathbb{C}_+}$,*

*then*

$$t \mapsto ST(t)R(\omega, A)x \in L^p(\mathbb{R}_+, X).$$

*If $\Omega = \Omega_M$ and, for some $\alpha, \beta > 0$,*

$$\|G(\lambda)\| \leq C(1 + |\operatorname{Im}\lambda|)^\alpha M(|\operatorname{Im}\lambda|)^\beta, \qquad \lambda \in \Omega_M,$$

*then, for some $k > 0$,*

$$t \mapsto M_{\log}^{-1}(kt)ST(t)R(\omega, A)x \in L^p(\mathbb{R}_+, X).$$

Newman-Korevaar approach to the Ingham-Karamata Theorem via contour integrals

Choose parameters of contour carefully as functions of $t$; use pointwise estimates for various terms

$p < \infty$: Use that arc-length on the contour in the left-half plane is a Carleson measure; obtain an $L^p$-estimate.

# Outline of proof: Hilbert space, polynomial case

Bounded semigroup on Hilbert space, $M(s) = C(1+s)^\alpha$.

- Use complex analysis to show

$$\|(\lambda - A)^{-1}(-A)^{-\alpha}\| \le C \quad (\operatorname{Re}\lambda \ge 0).$$

- Use Plancherel's Theorem to show

$$\|T(t)(-A)^{-\alpha}\| \le Ct^{-1}.$$

- Use semigroup property and interpolation (moment inequality) to show

$$\|T(t)(-A)^{-1}\| \le Ct^{-1/\alpha}.$$

# Outline of proof: Hilbert space, regularly varying case

Hilbert space, $M(s) \sim \frac{s^\alpha}{\ell(s)}$, $\ell$ increasing.

- Find a complete Bernstein function $f_\ell$, as large as possible, such that

$$\|(\lambda - A)^{-1} A^{-(\alpha-1)} f_\ell(-A^{-1})\| \le C \quad (\operatorname{Re}\lambda \ge 0).$$

  For example, $f_{\log}(-A^{-1}) = -A^{-1}\log(I - A)$.
- Use Plancherel's Theorem to show

$$\|T(t)A^{-(\alpha-1)} f_\ell(-A^{-1})\| \le Ct^{-1}.$$

- Use an interpolation inequality for complete Bernstein functions of semigroup generators, together with an Abelian/Tauberian theorem for Stieltjes transforms (Karamata, 1930s), to remove the log term in the $M_{\log}$ result.

# Example of $L^2$-version

$X$ Hilbert space, $T(t)$ contractions

Assume that $D(A) = D(A^*)$. Consider $-(A + A^*)$, symmetric, non-negative.

Let $S$ be any non-negative, self-adjoint extension of $-(A + A^*)$, $B = S^{1/2}$

**Theorem**

*Assume in addition that $\sigma(A) \cap i\mathbb{R}$ is empty, and $\|R(is, A)\| \le M(|s|)$. Then, for all $x \in X$,*

$$\int_0^\infty M_{\log}^{-1}(kt)^2 \|BT(t)A^{-1}x\|^2 \, dt < \infty.$$

The $M_{\log}$-theorem, or its improvements on Hilbert space, can be applied to damped wave equations whenever the resolvent of $A$ can be estimated. In particular,

$$\|(is - A)^{-1}\| = O(|s|^\alpha) \iff E(u,t) = O(t^{1/\alpha}).$$

The conditions of the earlier $L^2$-result hold.

**Corollary**

*Assume that the damped wave equation satisfies*

$$E(u,t) \leq r(t)^2 E(u,0)$$

*for all classical solutions $u$, and some decreasing function $r(t)$. Then*

$$\int_0^\infty \left| \frac{d}{dt} E(u,t) \right| M_{\log}^{-1}(kt)^2 \, dt < \infty$$

*where $M(s) = r^{-1}(c/s)$, for some $c, k > 0$.*

# Anosov flows

Let $\mathcal{M}$ be a (compact) Riemannian manifold (without boundary), and $\{\varphi_t : t \in \mathbb{R}\}$ be a smooth flow on $\mathcal{M}$. So-called *Anosov flows* have hyperbolic behaviour on the tangent spaces of $\mathcal{M}$, chaotic behaviour on $\mathcal{M}$, and mixing properties, i.e. for smooth $f, g$,

$$\int_{\mathcal{M}} f.(g \circ \varphi_t) \to \int_{\mathcal{M}} f \int_{\mathcal{M}} g \qquad (t \to \infty).$$

What is the rate of convergence?

Ruelle, Pollicott, Chernoff, Dolgopyat, Liverani, Tsujii, Butterley, Faure, Sjöstrand, Dyatlov, Zworski,.....

# Dolgopyat estimate

$$C^\infty(\mathcal{M}) \subset X \subset C(\mathcal{M})$$
$$T(t)f = f \circ \varphi_t, \qquad \text{generator } A$$

Ingredients:

1. Quasi-compactness argument for spectral gap
2. Resolvent estimate for large $|s|$

$$\left\|(\alpha + is - A)^{-\gamma \log |s|}\right\| \leq C(\lambda + \alpha)^{-\gamma \log |s|}. \qquad \text{(Dol)}$$

(fixed $\alpha, \gamma, \lambda, C > 0$ with $\gamma(\lambda + \alpha) < 1$)

(Dol) implies
$$\|(is - A)^{-1}\| = O(\log |s|)$$
and a polynomial bound for $-\lambda/2 < \operatorname{Re} z < 0$. Exponential rate of convergence follows from (an easy case of) the $M_{\log}$-theorem.