Spectral multipliers for generators of symmetric contraction semigroups

A. Carbonaro (U. of Genova)

Joint work with O. Dragičević (U. of Ljubljana)

Besançon, October 2014

Let (Ω, ν) be a σ-finite measure space, and let $\mathcal{A}$ be a nonnegative selfadjoint operator on $L^2(\Omega, \nu)$.

$$T_t := e^{-t\mathcal{A}} = \int_0^\infty e^{-t\lambda}\,\mathrm{d}\mathcal{P}_\lambda, \qquad t > 0.$$

We shall always assume that $(T_t)_{t>0}$ is a symmetric contraction semigroup:

$$\|T_t f\|_p \leqslant \|f\|_p \quad \forall f \in L^p(\Omega, \nu) \cap L^2(\Omega, \nu),$$

whenever $t > 0$ and $p \in [1, \infty]$.

- The semigroup $(T_t)_{t>0}$ is sub-Markovian if, in addition,

$$T_t f \geqslant 0, \quad \forall f \in L^2_+(\Omega, \nu).$$

- The semigroup $(T_t)_{t>0}$ is Markovian if it is sub-Markovian and

$$T_t 1 = 1, \quad t > 0$$

If $m \in L^\infty(\mathbb{R}_+)$, then by spectral theorem the operator

$$m(\mathcal{A})f = \int_0^\infty m(\lambda)\,\mathrm{d}\mathcal{P}_\lambda f, \quad f \in L^2(\Omega,\nu)$$

is bounded on $L^2(\Omega,\nu)$.

Definition

$$m \in \mathcal{M}_p(\mathcal{A}) \iff m(\mathcal{A}) \in \mathcal{B}(L^p(\Omega,\nu)), \quad 1 \leqslant p \leqslant \infty.$$

Multipliers problem

Given $p \in (1,\infty)$, characterize $\mathcal{M}_p(\mathcal{A})$.

Necessary conditions? Sufficient conditions?

A bit of notation

$$\mathbf{S}_\phi := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \phi\}, \quad 0 < \phi < \pi$$

$$H^\infty(\mathbf{S}_\phi) := \{\text{bounded and holomorphic } m : \mathbf{S}_\phi \to \mathbb{C}\}$$

Mihilin-Hörmander condition

$\psi : \mathbb{R} \to [0,1]$, smooth, supported in $(1/4, 4)$, $\psi = 1$ in $[1/2, 2]$.

$$H^\infty(\mathbf{S}_\phi; J) = \{m \in H^\infty(\mathbf{S}_\phi) : \|m\|_{\phi;J} < \infty\},$$

where $J \in \mathbb{R}_+$ and

$$\|m\|_{\phi;J} := \sup_{R>0} \left\|\psi(\cdot) m(e^{i\phi} R \cdot)\right\|_{H^J(\mathbb{R})} + \sup_{R>0} \left\|\psi(\cdot) m(e^{-i\phi} R \cdot)\right\|_{H^J(\mathbb{R})}$$

Finite dimensional Ornstein-Uhlenbeck operator

$\mathcal{L}_{ou} = \Delta + x \cdot \nabla$ on $\mathbb{R}^n$ endowed with the Gaussian measure

$$\phi_p^* := \arcsin \left| 1 - \frac{2}{p} \right|$$

Theorem (García-Cuerva, Mauceri, Meda, Sjögren, Torrea 2001)

If $m \in H^\infty(\mathbf{S}_{\phi_p^*}; J)$, $J > 5/2$, then

$$\|m(\mathcal{L}_{ou})\|_p \leqslant C(n, p) \left(\|m\|_{\phi_p^*; J} + |m(0)| \right)$$

Mauceri, Meda and Sjögren (2004): $J > 1$ suffices.

Finite dimensional Ornstein-Uhlenbeck operator

$\mathcal{L}_{ou} = \Delta + x \cdot \nabla$ on $\mathbb{R}^{n}$ endowed with the Gaussian measure

$$\phi_p^* := \arcsin \left| 1 - \frac{2}{p} \right|$$

Theorem (García-Cuerva, Mauceri, Meda, Sjögren, Torrea 2001)

If $m \in H^\infty(\mathbf{S}_{\phi_p^*}; J)$, $J > 5/2$, then

$$\|m(\mathcal{L}_{ou})\|_p \leqslant C(n, p) \left(\|m\|_{\phi_p^*;J} + |m(0)| \right)$$

Mauceri, Meda and Sjögren (2004): $J > 1$ suffices.

The angle ϕ_p^* is sharp (by Epperson theorem).

Finite dimensional Ornstein-Uhlenbeck operator

$\mathcal{L}_{ou} = \Delta + x \cdot \nabla$ on $\mathbb{R}^n$ endowed with the Gaussian measure

$$\phi_p^* := \arcsin \left| 1 - \frac{2}{p} \right|$$

Theorem (García-Cuerva, Mauceri, Meda, Sjögren, Torrea 2001)

If $m \in H^\infty(\mathbf{S}_{\phi_p^*}; J)$, $J > 5/2$, *then*

$$\|m(\mathcal{L}_{ou})\|_p \leqslant C(n,p) \left(\|m\|_{\phi_p^*;J} + |m(0)| \right)$$

Mauceri, Meda and Sjögren (2004): $J > 1$ *suffices.*

The angle ϕ_p^* is sharp (by Epperson theorem).

Theorem (Hebisch, Mauceri, Meda 2004)

$\sup_{t>0} \|m(t\mathcal{L}_{ou})\|_p < \infty \Rightarrow m \in H^\infty(\mathbf{S}_{\phi_p^*})$

Finite dimensional Ornstein-Uhlenbeck operator

$\mathcal{L}_{ou} = \Delta + x \cdot \nabla$ on $\mathbb{R}^n$ endowed with the Gaussian measure

$$\phi_p^* := \arcsin \left| 1 - \frac{2}{p} \right|$$

Theorem (García-Cuerva, Mauceri, Meda, Sjögren, Torrea 2001)

If $m \in H^\infty(\mathbf{S}_{\phi_p^*}; J)$, $J > 5/2$, then

$$\|m(\mathcal{L}_{ou})\|_p \leqslant C(n, p) \left(\|m\|_{\phi_p^*;J} + |m(0)| \right)$$

Mauceri, Meda and Sjögren (2004): $J > 1$ suffices.
The theorem does not apply to O-U operators on Wiener spaces

The angle ϕ_p^* is sharp (by Epperson theorem).

Theorem (Hebisch, Mauceri, Meda 2004)

$\sup_{t>0} \|m(t\mathcal{L}_{ou})\|_p < \infty \Rightarrow m \in H^\infty(\mathbf{S}_{\phi_p^*})$

The angle ϕ_p^* is related to all symmetric contraction semigroups:

$$\phi_p := \frac{\pi}{2} - \phi_p^*$$

Theorem (Kriegler 2011)

$(T_t)_{t>0}$ extends to an analytic contraction semigroup on $L^p(\Omega,\nu)$ in the sector $\mathbf{S}_{\phi_p}$:

$$\|T_z f\|_p \leqslant \|f\|_p \quad \forall f \in L^p(\Omega,\nu), \quad \forall z \in \mathbf{S}_{\phi_p}$$

- Proved by Bakry (1989) for symmetric diffusion semigroups
- Proved by Liskevich and Perelmuter (1995) for sub-Markovian semigroups

The angle ϕ_p^* is related to all symmetric contraction semigroups:

$$\phi_p := \frac{\pi}{2} - \phi_p^*$$

Theorem (Kriegler 2011)

$(T_t)_{t>0}$ extends to an analytic contraction semigroup on $L^p(\Omega, \nu)$ in the sector $\mathbf{S}_{\phi_p}$:

$$\|T_z f\|_p \leqslant \|f\|_p \quad \forall f \in L^p(\Omega, \nu), \quad \forall z \in \mathbf{S}_{\phi_p}$$

- Proved by Bakry (1989) for symmetric diffusion semigroups
- Proved by Liskevich and Perelmuter (1995) for sub-Markovian semigroups

A natural question: sharp universal multiplier theorem

$$H^\infty(\mathbf{S}_{\phi_p^*}; J) \subset \mathcal{M}_p(\mathcal{A}) \qquad \forall \text{ g.s.c. } \mathcal{A} \text{ ?}$$

Universal multiplier theorems

For $M \in L^\infty(\mathbb{R}_+)$, define its Laplace transform $\widetilde{M}$ by

$$\widetilde{M}(\lambda) := \lambda \int_0^\infty M(t) e^{-t\lambda} \, \mathrm{d}t$$

Theorem (Stein 1970)

Suppose that $\mathcal{A}$ is a Markovian generator and $p > 1$. Then

$$\widetilde{L^\infty(\mathbb{R}_+)} \subset \mathcal{M}_p(\mathcal{A})$$

In particular, for all $\phi > \pi/2$,

$$H^\infty(\mathbf{S}_\phi) \subset \mathcal{M}_p(\mathcal{A})$$

Proof. Abstract Littlewood-Paley theory: boundedness of vertical square funtion via martingales theory

Theorem (Cowling 1983)

Let $\phi_p^C = \pi|1/2 - 1/p|$. *Then for all* $\phi > \phi_p^C$,

$$H^\infty(\mathbf{S}_\phi) \subset \mathcal{M}_p(\mathcal{A})$$

Proof. Coifman-Weiss transference and complex interpolation.

Theorem (Kunstmann and Štrkalj 2003)

Let $\mathcal{A}$ *be sub-Markovian. Then* $\exists\ \phi_p^{K\check{S}} < \phi_p^C$ *such that, for all* $\phi > \phi_p^{K\check{S}}$

$$H^\infty(\mathbf{S}_\phi) \subset \mathcal{M}_p(\mathcal{A})$$

Proof. Analyticity, Rademacher boundedness, interpolation.

Theorem (Kriegler 2011)

The condition that $\mathcal{A}$ *is sub-Markovian can be removed in the theorem above*

Note that $0 < \phi_p^* < \phi_p^{K\check{S}} < \phi_p^C$, $p \neq 2$.

Theorem (C., Dragičević)

Suppose that $J > 3/2$ and $p > 1$. Then, $\forall$ g.s.c. $\mathcal{A}$,

$$H^\infty(\phi_p^*; J) \subset \mathcal{M}_p(\mathcal{A}),$$

and

$$\|m(\mathcal{A})\|_p \lesssim_J C_0(p) \left(\|m\|_{\phi_p^*;J} + |m(0)|\right).$$

The angle ϕ_p^* is sharp.

- The theorem in particular applies to $\mathcal{L}_{ou}$, as well as to O-U operators on Wiener spaces
- However for $\mathcal{L}_{ou}$, $J > 1$ suffices.
- We have $C_0(p) \sim p^{9/4} \log p$, for $p > 2$.

Theorem (Meda 1990)

It is enough to prove

$$\|\mathcal{A}^{is} f\|_p \lesssim_p (1+|s|)^{1/2} e^{\phi_p^*|s|} \|f\|_p, \quad \forall f \in \overline{\mathrm{R}(\mathcal{A}_p)} \quad \forall s \in \mathbb{R}_-$$

Theorem (Meda 1990)

It is enough to prove

$$\|\mathcal{A}^{is} f\|_p \lesssim_p (1+|s|)^{1/2} e^{\phi_p^* |s|} \|f\|_p, \quad \forall f \in \overline{\mathrm{R}(\mathcal{A}_p)} \quad \forall s \in \mathbb{R}_-$$

$$\gamma_\phi := \{te^{i\phi} : t > 0\} \quad \text{and} \quad \phi^* := \pi/2 - \phi$$

For $f \in \mathrm{R}(\mathcal{A}_2) \cap \mathrm{D}(\mathcal{A}_2)$,

$$\mathcal{A}^{is} f = \frac{2^{1-is}}{\Gamma(1-is)} \int_{\gamma_\phi} z^{-is} \mathcal{A} T_{2z} f \, \mathrm{d}z, \quad \phi \in [0, \pi/2)$$

Stirling's formula gives

$$\left| \int_\Omega \mathcal{A}^{is}(f) \overline{g} \, \mathrm{d}\nu \right| \lesssim (1+|s|)^{-1/2} e^{\phi^* |s|} \int_{\gamma_\phi} \left| \int_\Omega \mathcal{A}(T_z f) \overline{T_{\overline{z}} g} \, \mathrm{d}\nu \right| \mathrm{d}|z|$$

For $p \in (2, \infty)$ set $q := p/(p-1)$

Bilinear embedding (C., Dragičević)

Suppose that $\phi \in (-\phi_p, \phi_p)$. Then,

$$\int_{\gamma_\phi} \left| \int_\Omega \mathcal{A}(T_z f)\overline{T_{\bar{z}} g}\, \mathrm{d}\nu \right| \mathrm{d}|z| \leqslant \frac{C_1(p)}{|\phi_p - \phi|} \left(\|f\|_p^p + \|g\|_q^q \right),$$

for all $f \in L^p(\Omega, \nu)$ and $g \in L^q(\Omega, \nu)$.

- By replacing f and g with λf and $\lambda^{-1} g$, $\lambda > 0$, and optimizing in λ, we can replace we right hand side by

$$\frac{C_2'(p)}{|\phi_p - \phi|} \|f\|_p \|g\|_q$$

- To prove the multiplier theorem, choose ϕ s.t. $|\phi_p - \phi| \sim (1 + |s|)^{-1}$

Bellman function and complex time “heat” flow

If $Q : \mathbb{R}^4 \to \mathbb{R}_+$, consider the flow

$$\mathcal{E}(t) := \int_\Omega Q\left(T_{te^{i\phi}} f,\, T_{te^{-i\phi}} g\right) \mathrm{d}\nu, \quad t \geqslant 0.$$

We are interested in finding a function Q, possibly depending on ϕ and p, whose corresponding flow is “regular”, and

$$\mathcal{E}(0) \leqslant A_0(\|f\|_p^p + \|g\|_q^q) \tag{1}$$

$$-\mathcal{E}'(t) \geqslant B_0 \left| \int_\Omega \mathcal{A}(T_{te^{i\phi}} f)\, \overline{T_{te^{-i\phi}} g} \, \mathrm{d}\nu \right| \tag{2}$$

If this is the case, by integrating both sides of (2) from 0 to ∞, we obtain the desired bilinear embedding.

$$Q(\zeta,\eta) \leqslant A_0 \left(|\zeta|^p + |\eta|^q\right), \quad \forall \zeta,\eta \in \mathbb{R}^2 \;\Rightarrow\; (1)$$

$$Q(\zeta,\eta) \leqslant A_0\left(|\zeta|^p + |\eta|^q\right), \quad \forall \zeta,\eta \in \mathbb{R}^2 \ \Rightarrow \ (1)$$

$$\partial_\zeta := \partial_{\zeta_1} - i\partial_{\zeta_2} \quad \text{and} \quad \partial_\eta := \partial_{\eta_1} - i\partial_{\eta_2}$$

Since $T_{te^{\pm i\phi}} f \in \mathrm{D}(\mathcal{A}_p)$ and $T_{te^{\pm i\phi}} g \in \mathrm{D}(\mathcal{A}_q)$, condition (2) reduces to:

$$\left|\int_\Omega \mathcal{A}(f)\overline{g}\,\mathrm{d}\nu\right| \lesssim \int_\Omega \Re\left(e^{i\phi}\partial_\zeta Q(f,g)\mathcal{A}f + e^{-i\phi}\partial_\eta Q(f,g)\mathcal{A}g\right)\,\mathrm{d}\nu,$$

for all $f \in \mathrm{D}(\mathcal{A}_p)$, $g \in \mathrm{D}(\mathcal{A}_q)$

Pointwise conditions on second-order partial derivatives of Q such that the inequality above holds?

$$Q(\zeta,\eta) \leqslant A_0\left(|\zeta|^p + |\eta|^q\right), \quad \forall \zeta,\eta \in \mathbb{R}^2 \;\Rightarrow\; (1)$$

$$\partial_\zeta := \partial_{\zeta_1} - i\partial_{\zeta_2} \quad \text{and} \quad \partial_\eta := \partial_{\eta_1} - i\partial_{\eta_2}$$

Since $T_{te^{\pm i\phi}} f \in \mathrm{D}(\mathcal{A}_p)$ and $T_{te^{\pm i\phi}} g \in \mathrm{D}(\mathcal{A}_q)$, condition (2) reduces to:

$$\int_M |\,\mathrm{d}f|\,|\,\mathrm{d}g|\;\mathrm{d}\mu \lesssim \int_M \Re\left(e^{i\phi}\partial_\zeta Q(f,g)\Delta f + e^{-i\phi}\partial_\eta Q(f,g)\Delta g\right)\,\mathrm{d}\mu,$$

for all $f,g \in C_c^\infty(M)$

Pointwise conditions on second-order partial derivatives of Q such that the inequality above holds?

We first consider the special case when $\mathcal{A} = \Delta$ is the Laplace-Beltrami operator on a complete manifold (M,μ).

$\mathcal{O}_\phi :=$ Rotation of angle ϕ in $\mathbb{R}^2$, $\qquad H(Q) :=$ Hessian of Q

$$\mathcal{R}_\phi(Q) := \frac{1}{2}\left(\begin{bmatrix} \mathcal{O}_\phi^T & 0 \\ 0 & \mathcal{O}_{-\phi}^T \end{bmatrix} \cdot \mathrm{H}(Q) + \mathrm{H}(Q) \cdot \begin{bmatrix} \mathcal{O}_\phi & 0 \\ 0 & \mathcal{O}_{-\phi} \end{bmatrix}\right)$$

An integration by parts ($f, g \in C_c^\infty(M)$) gives

$$\int_M \Re\left(e^{i\phi}\partial_\zeta Q(f,g)\Delta f + e^{-i\phi}\partial_\eta Q(f,g)\Delta g\right)\,\mathrm{d}\mu$$
$$= \int_M \mathcal{R}_\phi(Q)\left[(f,g);(\,\mathrm{d}f,\,\mathrm{d}g)\right]\,\mathrm{d}\mu$$

$\mathcal{O}_\phi :=$ Rotation of angle ϕ in $\mathbb{R}^2$, $\qquad H(Q) :=$ Hessian of Q

$$\mathcal{R}_\phi(Q) := \frac{1}{2}\left(\begin{bmatrix} \mathcal{O}_\phi^T & 0 \\ 0 & \mathcal{O}_{-\phi}^T \end{bmatrix} \cdot \mathrm{H}(Q) + \mathrm{H}(Q) \cdot \begin{bmatrix} \mathcal{O}_\phi & 0 \\ 0 & \mathcal{O}_{-\phi} \end{bmatrix}\right)$$

An integration by parts ($f, g \in C_c^\infty(M)$) gives

$$\int_M \Re\left(e^{i\phi}\partial_\zeta Q(f,g)\Delta f + e^{-i\phi}\partial_\eta Q(f,g)\Delta g\right) \mathrm{d}\mu$$
$$= \int_M \mathcal{R}_\phi(Q)\left[(f,g);(\mathrm{d}f,\, \mathrm{d}g)\right] \mathrm{d}\mu$$

Therefore, at least in this case, the monotonicity property of $\mathcal{E}$ holds, provided that Q satisfies

$$\mathcal{R}_\phi(Q)[\xi;(\omega_1,\omega_2)] \geqslant B_0\left(\tau(\xi)|\omega_1|^2 + \tau(\xi)^{-1}|\omega_2|^2\right),$$

for all $\xi \in \mathbb{R}^4$, $\omega_1, \omega_2 \in \mathbb{R}^2$.

The heat-flow method reduces the proof of the bilinear embedding to finding, for every $p > 2$, $\phi \in (-\phi_p, \phi_p)$ a function Q s.t.

(a) The associated flow $\mathcal{E}$ is regular (estimates of $\partial_\zeta Q$ and $\partial_\eta Q$)

(b) $Q(\zeta, \eta) \lesssim_{p,\phi} (|\zeta|^p + |\eta|^q)\,, \quad \forall \zeta, \eta \in \mathbb{R}^2$

(c) There exists $\tau : \mathbb{R}^4 \to (0, \infty)$ such that

$$\mathcal{R}_\phi(Q)[\xi; (\omega_1, \omega_2)] \gtrsim_{p,\phi} \left(\tau(\xi)|\omega_1|^2 + \tau(\xi)^{-1}|\omega_2|^2\right),$$

for all $\xi \in \mathbb{R}^4$ and every $\omega_1, \omega_2 \in \mathbb{R}^2$.

- Note that $\mathcal{R}_0(Q) = \mathrm{H}(Q)$.
- When $\phi = 0$ similar heat-flow techniques where previously employed (for different problems) in the Euclidean case by Dragičević, Nazarov, Petermichl, Volberg and in the Riemannian setting by myself and Dragičević.

Nazarov-Treil Bellman function: Fix $p > 2$ and $\delta > 0$

$$Q(\zeta,\eta) := |\zeta|^p + |\eta|^q + \delta \begin{cases} |\zeta|^2|\eta|^{2-q} & ; \quad |\zeta|^p \leqslant |\eta|^q \\ \dfrac{2}{p}|\zeta|^p + \left(\dfrac{2}{q} - 1\right)|\eta|^q & ; \quad |\zeta|^p \geqslant |\eta|^q \end{cases}$$

- $Q \in C^1(\mathbb{R}^4)$ and $Q \in C^2(\mathbb{R}^4 \setminus \Upsilon_0)$, where

$$\Upsilon_0 = \{(\zeta,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2\,;(\eta = 0) \vee (|\zeta|^p = |\eta|^q)\}$$

- $Q(\zeta,\eta) \lesssim_\delta (|\zeta|^p + |\eta|^q)$
- $|\partial_\zeta Q(\zeta,\eta)| \lesssim_{p,\delta} \max\{|\zeta|^{p-1}, |\eta|\}, \quad |\partial_\eta Q(\zeta,\eta)| \lesssim_{q,\delta} |\eta|^{q-1}$

For $0 < \varepsilon < 1/2$, set

$$p_\varepsilon := (p - 2\varepsilon)/(1 - \varepsilon), \qquad \delta := \delta(p, \varepsilon) = \varepsilon/44p.$$

Note that $\phi_{p_\varepsilon} < \phi_p < \phi_{p_\varepsilon} + 2\varepsilon(p-2)/p\sqrt{p-1}$

Theorem (C., Dragičević)

If $\phi \in [-\phi_{p_\varepsilon}, \phi_{p_\varepsilon}]$*, then*

$$\mathcal{R}_\phi(Q)[\xi; (\omega_1, \omega_2)] \geqslant \varepsilon \frac{\cos\phi}{22p} |\omega_1||\omega_2|,$$

for all $\xi \in \mathbb{R}^4 \setminus \Upsilon_0$*, and* $\omega_1, \omega_2 \in \mathbb{R}^2$.

For $0 < \varepsilon < 1/2$, set

$$p_\varepsilon := (p - 2\varepsilon)/(1 - \varepsilon), \qquad \delta := \delta(p, \varepsilon) = \varepsilon/44p.$$

Note that $\phi_{p_\varepsilon} < \phi_p < \phi_{p_\varepsilon} + 2\varepsilon(p-2)/p\sqrt{p-1}$

Theorem (C., Dragičević)

If $\phi \in [-\phi_{p_\varepsilon}, \phi_{p_\varepsilon}]$, then

$$\mathcal{R}_\phi(Q * \psi_r)[\xi; (\omega_1, \omega_2)] \geqslant \varepsilon \frac{\cos\phi}{22p} \, |\omega_1||\omega_2| \, ,$$

for all $\xi \in \mathbb{R}^4$, and $\omega_1, \omega_2 \in \mathbb{R}^2$.

Where ψ_r, $r > 0$, is a smooth nonnegative compactly supported approximation of the identity in $\mathbb{R}^4$

For $0 < \varepsilon < 1/2$, set

$$p_\varepsilon := (p - 2\varepsilon)/(1-\varepsilon), \qquad \delta := \delta(p,\varepsilon) = \varepsilon/44p.$$

Note that $\phi_{p_\varepsilon} < \phi_p < \phi_{p_\varepsilon} + 2\varepsilon(p-2)/p\sqrt{p-1}$

Theorem (C., Dragičević)

If $\phi \in [-\phi_{p_\varepsilon}, \phi_{p_\varepsilon}]$, then

$$\mathcal{R}_\phi(Q * \psi_r)[\xi; (\omega_1, \omega_2)] \geqslant \varepsilon \frac{\cos\phi}{22p} \left(\tau(\xi)|\omega_1|^2 + \tau^{-1}(\xi)|\omega_2|^2\right),$$

for all $\xi \in \mathbb{R}^4$, and $\omega_1, \omega_2 \in \mathbb{R}^2$.

Where ψ_r, $r > 0$, is a smooth nonnegative compactly supported approximation of the identity in $\mathbb{R}^4$

For $0 < \varepsilon < 1/2$, set

$$p_\varepsilon := (p - 2\varepsilon)/(1 - \varepsilon), \qquad \delta := \delta(p, \varepsilon) = \varepsilon/44p.$$

Note that $\phi_{p_\varepsilon} < \phi_p < \phi_{p_\varepsilon} + 2\varepsilon(p-2)/p\sqrt{p-1}$

Theorem (C., Dragičević)

If $\phi \in [-\phi_{p_\varepsilon}, \phi_{p_\varepsilon}]$, then

$$\mathcal{R}_\phi(Q)[\xi; (\omega_1, \omega_2)] \geqslant \varepsilon \frac{\cos\phi}{22p} \, |\omega_1||\omega_2| \, ,$$

for all $\xi \in \mathbb{R}^4 \setminus \Upsilon_0$, and $\omega_1, \omega_2 \in \mathbb{R}^2$.

The theorem was already known for $\phi = 0$ (Nazarov and Treil), and it was very purpose of constructing the prototype of Q.

The theorem can be considered as a natural extension of a convexity result proved by Bakry for studying analyticity of diffusion semigroups

$$F_p(\zeta) := |\zeta|^p, \qquad \zeta \in \mathbb{R}^2$$

Lemma (Bakry 1989)

The matrix

$$\left(\mathcal{O}_\phi^T \cdot \mathrm{H}(F_p) + \mathrm{H}(F_p) \cdot \mathcal{O}_\phi\right)$$

define a nonnegative quadratic form on $\mathbb{R}^2$ *iff* $\phi \in [-\phi_p, \phi_p]$.

Bilinear embedding in the general case: We reduced to prove

$$A_0 \left| \int_\Omega \mathcal{A}(f)\overline{g}\,\mathrm{d}\nu \right| \leqslant \int_\Omega \Re\left(e^{i\phi}\partial_\zeta Q(f,g)\mathcal{A}f + e^{-i\phi}\partial_\eta Q(f,g)\mathcal{A}g\right)\mathrm{d}\nu, \tag{*}$$

for all $f \in \mathrm{D}(\mathcal{A}_p)$, $g \in \mathrm{D}(\mathcal{A}_q)$, and $\phi \in [-\phi_{p_\varepsilon}, \phi_{p_\varepsilon}]$.

Bilinear embedding in the general case: We reduced to prove

$$A_0\left|\int_\Omega \mathcal{A}(f)\overline{g}\,\mathrm{d}\nu\right| \leqslant \int_\Omega \Re\left(e^{i\phi}\partial_\zeta Q(f,g)\mathcal{A}f + e^{-i\phi}\partial_\eta Q(f,g)\mathcal{A}g\right)\mathrm{d}\nu, \tag{*}$$

for all $f \in \mathrm{D}(\mathcal{A}_p)$, $g \in \mathrm{D}(\mathcal{A}_q)$, and $\phi \in [-\phi_{p_\varepsilon}, \phi_{p_\varepsilon}]$.

(i) ineq. (*) is true for the heat generator on the two-point space:

$$\mathcal{A} = \mathcal{G} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

on $\mathbb{C}^2 = L^\infty(\{a,b\},\nu_{a,b})$, $\nu_{a,b} = (\delta_a + \delta_b)/2$

Bilinear embedding in the general case: We reduced to prove

$$A_0\left|\int_\Omega \mathcal{A}(f)\overline{g}\,\mathrm{d}\nu\right| \leqslant \int_\Omega \Re(e^{i\phi}\partial_\zeta Q(f,g)\mathcal{A}f + e^{-i\phi}\partial_\eta Q(f,g)\mathcal{A}g)\,\mathrm{d}\nu, \tag{*}$$

for all $f \in \mathrm{D}(\mathcal{A}_p)$, $g \in \mathrm{D}(\mathcal{A}_q)$, and $\phi \in [-\phi_{p_\varepsilon}, \phi_{p_\varepsilon}]$.

(i) ineq. (*) is true for the heat generator on the two-point space:

$$\mathcal{A} = \mathcal{G} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

on $\mathbb{C}^2 = L^\infty(\{a,b\}, \nu_{a,b})$, $\nu_{a,b} = (\delta_a + \delta_b)/2$

$$2\int_{\{a,b\}} \mathcal{G}u \cdot \overline{v}\,\mathrm{d}\nu_{a,b} = (u(a)-u(b)) \cdot \overline{(v(a)-v(b))},$$

Ineq. (*) for $\mathcal{G}$ follows by the mean value theorem and our theorem about $\mathcal{R}_\phi(Q)$.

Bilinear embedding in the general case: We reduced to prove

$$A_0\left|\int_\Omega (I - T_t)(f)\overline{g}\,\mathrm{d}\nu\right| \leqslant \int_\Omega \Re(e^{i\phi}\partial_\zeta Q(f,g)(I - T_t)f + \cdots)\,\mathrm{d}\nu, \quad (*)$$

for all $f \in \mathrm{D}(\mathcal{A}_p)$, $g \in \mathrm{D}(\mathcal{A}_q)$, and $\phi \in [-\phi_{p_\varepsilon}, \phi_{p_\varepsilon}]$.

(i) ineq. (*) is true for the heat generator on the two-point space:

$$\mathcal{A} = \mathcal{G} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

on $\mathbb{C}^2 = L^\infty(\{a,b\}, \nu_{a,b})$, $\nu_{a,b} = (\delta_a + \delta_b)/2$

$$2\int_{\{a,b\}} \mathcal{G}u \cdot \overline{v}\,\mathrm{d}\nu_{a,b} = (u(a) - u(b)) \cdot \overline{(v(a) - v(b))},$$

Ineq. (*) for $\mathcal{G}$ follows by the mean value theorem and our theorem about $\mathcal{R}_\phi(Q)$.

(ii) We can assume that $\nu(\Omega) < \infty$. Then, in the Markovian case we have the following representation formula:

Let $\hat{}: L^\infty(\Omega, \nu) \to C(\hat{\Omega})$ be the Gelfand isomorphism, where $\hat{\Omega}$ denotes the maximal ideal space of the C^*-algebra $L^\infty(\Omega, \nu)$.

Then there exists a positive symmetric Radon measure m_t on $\hat{\Omega} \times \hat{\Omega}$ s.t.

$$\int_\Omega (I - T_t)(u)\overline{v}\, \mathrm{d}\nu = \int_{\hat{\Omega}\times\hat{\Omega}} \left(\int_{\{x,y\}} \mathcal{G}\hat{u} \cdot \overline{\hat{v}}\, \mathrm{d}\nu_{x,y} \right) \mathrm{d}m_t(x, y),$$

for all $u, v \in L^\infty(\Omega, \nu)$.

Therefore, in the Markovian case, (*) follows by combining (i) with the above rep. formula

(iii) In the general case we have to adapt the representation formula