Improved Sobolev Inequalities, semi-groups and stratified Lie groups

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Classical Sobolev Inequalities ($\sim$ 1938)

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$$\|f\|_{L^q} \leq C\|(-\Delta)^{\frac{1}{2}} f\|_{L^p}$$

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Two **very different** *inequalities !*

Why so different ?

$\Longrightarrow$ If $1 < p < +\infty$:

$$\|(-\Delta)^{\frac{1}{2}} f\|_{L^p} \simeq \|\nabla f\|_{L^p}$$

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$\Longrightarrow$ If $1 = p$:

$$\|(-\Delta)^{\frac{1}{2}} f\|_{L^1} \neq \|\nabla f\|_{L^1}$$

less tools at hand...

Improved Sobolev Inequalities (Gérard, Meyer, Oru $\sim$ 1997)

Theorem

For a function $\nabla f \in L^p(\mathbb{R}^n)$ *and* $f \in \dot{B}_\infty^{-\beta,\infty}(\mathbb{R}^n)$

$$\|f\|_{L^q} \leq C\|\nabla f\|_{L^p}^{\theta}\|f\|_{\dot{B}_\infty^{-\beta,\infty}}^{1-\theta}$$

where $1 < p < q < +\infty$, $\theta = p/q$, $\theta = (1-\theta)\beta$.

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If $q = \frac{np}{n-p}$ (Sobolev), we have $L^q \subset \dot{B}_{\infty}^{-\beta,\infty}$ and

$$\begin{aligned}\|f\|_{L^q} \leq \|\nabla f\|_{L^p}^{\theta}\|f\|_{\dot{B}_{\infty}^{-\beta,\infty}}^{1-\theta} &= \|\nabla f\|_{L^p}\left(\frac{\|f\|_{\dot{B}_{\infty}^{-\beta,\infty}}}{\|\nabla f\|_{L^p}}\right)^{1-\theta} \leq \|\nabla f\|_{L^p}\left(\frac{C\|f\|_{L^q}}{\|\nabla f\|_{L^p}}\right)^{1-\theta} \\ &\leq c'\|\nabla f\|_{L^p}\end{aligned}$$

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$\Longrightarrow$ The case $p=1$ **can not** be treated by this method

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We will see here how to improve these inequalities in the setting of Stratified Lie groups

The structure of the Heisenberg group

- Consider $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the non commutative group law

$$x \cdot y = (x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - y_1 x_2)\right).$$

- Define δ_α for $\alpha > 0$ by

$$\begin{array}{rcl} \delta_\alpha : \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ x = (x_1, x_2, x_3) & \longmapsto & \delta_\alpha[x] = (\alpha x_1, \alpha x_2, \alpha^2 x_3) \end{array}$$

Topological dimension $n = 3$, Homogeneous dimension $N = 4$.

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- Norm : $\|x\| = \Big[(x_1^2 + x_2^2)^2 + 16 x_3^2\Big]^{\frac{1}{4}}$
- Distance : $d(x, y) = \|y^{-1} \cdot x\|$
- Haar measure = Lebesgue measure

Vector fields

We have a Lie algebra $\mathfrak{h}$ given by the vector fields

$$X_1 = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2\frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1\frac{\partial}{\partial x_3} \quad \text{and} \quad T = \frac{\partial}{\partial x_3}$$

and we have the identities

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- X_1 and X_2 are *homogeneous* of degree 1
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$$X_1\big(f(\delta_\alpha[x])\big) = \alpha(X_1 f)(\delta_\alpha[x]), \qquad X_2\big(f(\delta_\alpha[x])\big) = \alpha(X_2 f)(\delta_\alpha[x]),$$

$$\text{and} \qquad T\big(f(\delta_\alpha[x])\big) = \alpha^{\mathbf{2}}(Tf)(\delta_\alpha[x]).$$

- We define a gradient :

$$\nabla = (X_1, X_2)$$

- We define a Laplacian by the formula

$$\mathcal{J} = -({X_1}^2 + {X_2}^2)$$

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- in particular we can define :

$$\mathcal{J}^s = \int_0^{+\infty} \lambda^s dE(\lambda)$$

If $s < 0$ we have

$$\mathcal{J}^{\frac{-s}{2}} f(x) = C \int_0^{+\infty} t^{\frac{s}{2}-1} H_t f(x) dt$$

Proposition

Let $k \in \mathbb{N}$ *and* $m \in \mathcal{C}^k(\mathbb{R}^+)$, *with* $\|m\|_{(k)} = \sup_{\substack{0 \leq r \leq k \\ \lambda > 0}} (1+\lambda)^k |m^{(r)}(\lambda)|$.

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$$\left\| X^I M_t(\cdot) \right\|_{L^p} \leq C\, t^{-|I|/2 - N/2p'} \|m\|_{(k)}$$

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$\Longrightarrow$ $m(t\mathcal{J}) = e^{-t\mathcal{J}} = H_t$ is the **heat semi-group**.

- **Lebesgue spaces** $L^p(\mathbb{G})$.
- **weak-L^p spaces** : $\|f\|_{L^{p,\infty}} = \sup_{\sigma>0}\{\sigma\ |\{x\in\mathbb{G} : |f(x)| > \sigma\}|^{1/p}\}$.
- **Sobolev spaces**

$$\|f\|_{\dot{W}^{s,p}} = \|\mathcal{J}^{s/2}f\|_{L^p} \quad (1 < p < +\infty) \qquad\qquad \|f\|_{\dot{W}^{1,1}} = \|\nabla f\|_{L^1}$$

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- **Besov spaces** $\dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$.

$$\|f\|_{\dot{B}^{-\beta,\infty}_{\infty}} = \sup_{t>0}\ t^{\beta/2}\|H_t f\|_{L^\infty}$$

Theorem

Let $\mathbb{G}$ *be a stratified Lie group. If* $f \in \dot{W}^{s_1,p}(\mathbb{G})$ *and* $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$ *then*

$$\|f\|_{\dot{W}^{s,q}} \leq C\|f\|^{\theta}_{\dot{W}^{s_1,p}}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}$$

where $1 < p < q < +\infty$, $\theta = p/q$, $s = \theta s_1 - (1-\theta)\beta$ *and* $-\beta < s < s_1$.

Proof. We can rewrite this inequality in the following form

$$\|\mathcal{J}^{\frac{-(s_1-s)}{2}} f\|_{L^q} \leq C\|f\|_{L^p}^{\theta}\|f\|_{\dot{B}_{\infty}^{-\beta-s_1,\infty}}^{1-\theta}$$

$\Longrightarrow$ Using the Laplacian negative powers characterization we have

$$\mathcal{J}^{\frac{-(s_1-s)}{2}} f(x) = C\left(\int_0^T t^{\frac{s_1-s}{2}-1} H_t f(x) dt + \int_T^{+\infty} t^{\frac{s_1-s}{2}-1} H_t f(x) dt\right)$$

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$\Longrightarrow$ Set $T = \left(\|f\|_{\dot{B}_{\infty}^{-\beta-s_1,\infty}}/|\mathcal{M}f(x)|\right)^{\frac{2}{\beta+s_1}}$ and then

$$|\mathcal{J}^{\frac{-(s_1-s)}{2}} f(x)| \leq C_3|\mathcal{M}f(x)|^{\theta}\|f\|_{\dot{B}_{\infty}^{-\beta-s_1,\infty}}^{1-\theta}$$

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- *[Weak inequalities]*

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where $1 < q < +\infty$, $0 < s < 1/q < 1$, $\theta = 1/q$ and $\beta = \frac{1-sq}{q-1}$.

- *[Strong inequalities]*

$$\|f\|_{L^q} \leq C\|\nabla f\|^{\theta}_{L^1}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}$$

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Proposition (Modified Poincaré pseudo-inequality ($0 \leq s < 1$))

$$\|\mathcal{J}^{s/2}f - H_t\mathcal{J}^{s/2}f\|_{L^1} \leq C\ t^{\frac{1-s}{2}}\|\nabla f\|_{L^1}.$$

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$$I(f) = (\mathcal{J}^{s/2} f - H_t \mathcal{J}^{s/2} f)(x) = \left(\int_0^{+\infty} m(t\lambda) dE_\lambda\right) t^{1-s/2} \mathcal{J} f(x),$$

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- we cut this function $m(\lambda) = m_0(\lambda) + m_1(\lambda)$

$$\begin{aligned} I(f) &= \left(\int_0^{+\infty} m_0(t\lambda) dE_\lambda\right) t^{1-s/2} \mathcal{J} f(x) + \left(\int_0^{+\infty} m_1(t\lambda) dE_\lambda\right) t^{1-s/2} \mathcal{J} f(x) \\ &= t^{1-s/2} \mathcal{J} f * M_t^{(0)}(x) + t^{1-s/2} \mathcal{J} f * M_t^{(1)}(x) \end{aligned}$$

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with $m(\lambda) = \lambda^{s/2-1}(1 - e^{-\lambda})$ for $\lambda > 0$.

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$$\begin{aligned} I(f) &= \left(\int_0^{+\infty} m_0(t\lambda) dE_\lambda\right) t^{1-s/2} \mathcal{J} f(x) + \left(\int_0^{+\infty} m_1(t\lambda) dE_\lambda\right) t^{1-s/2} \mathcal{J} f(x) \\ &= t^{1-s/2} \mathcal{J} f * M_t^{(0)}(x) + t^{1-s/2} \mathcal{J} f * M_t^{(1)}(x) \\ &= t^{1-s/2} \nabla f * \nabla M_t^{(0)}(x) + t^{1-s/2} \nabla f * \nabla M_t^{(1)}(x) \end{aligned}$$

- taking the L^1 norm

$$\|I(f)\|_{L^1} \leq t^{1-s/2} \|\nabla f\|_{L^1} \|\nabla M_t^{(0)}\|_{L^1} + t^{1-s/2} \|\nabla f\|_{L^1} \|\nabla M_t^{(1)}\|_{L^1}$$

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- If $t_\alpha = \alpha^{-\left(\frac{2}{\beta+s}\right)}$, we obtain $\|H_{t_\alpha}\mathcal{J}^{s/2}f\|_{L^\infty} \leq \alpha$.

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$$\iff \|\mathcal{J}^{s/2} f\|_{L^{q,\infty}}^q \leq C \|\nabla f\|_{L^1}$$

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When $s = 0$ in the weak inequalities it is possible to obtain stronger estimations.

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$$A_\alpha = \{|f_\alpha| > 4\alpha\} \qquad B_\alpha = \{|f_\alpha - H_{t_\alpha}(f_\alpha)| > \alpha\}$$

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How to obtain strong inequalities from weak ones ?

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A first step : $\|f\|_{\dot{W}^{s,q}_{r}}$ with $q < r$?

$\Longrightarrow$ Non-local objects : $\mathcal{J}^{s/2}$

$\Longrightarrow$ Definition of Sobolev-Lorentz spaces (distribution function)

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