Hardy spaces on graphs, application to Riesz tranforms.

Joseph Feneuil
University of Grenoble

Besançon October 3rd, 2014

Definition of graphs

A graph is given by

- an countable (infinite) set Γ,
- a symmetric subset E of $\Gamma \times \Gamma$ called the set of edges of Γ.
  It induces a distance on Γ : if $(x, y) \in E$, then $d(x, y) = 1$.
  If B is a ball, then $C_j(B) = 2^{j+1}B \backslash 2^j B$ if $j \geq 2$ and $C_1(B) = 4B$.
- a measure m on Γ.
  We use $V(B)$ for the volume of the ball B.

Assumption : doubling volume property.

There exists $C > 0$ such that for all balls $B \subset \Gamma$, $V(2B) \leq CV(B)$.
Therefore, there exist $C > 0$ and $d > 0$ such that

$$V(\lambda B) \leq C\lambda^d V(B).$$

d_0 is the infimum of the possible d.

Markov operator

- P is a random walk (of step 1) reversible w.r.t the measure m. $p(x,y)$ is its kernel.
- The laplacian is defined by $\Delta = I - P$,
- The length of the gradient is defined by

$$\nabla f(x) = \left(\frac{1}{2} \sum_{y \in \Gamma} p(x,y) |f(x) - f(y)|^2 \right)^{\frac{1}{2}}.$$

Notice that $\|\nabla f\|_{L^2}^2 = \langle \Delta f, f \rangle = \|\Delta^{\frac{1}{2}} f\|_{L^2}^2$.

Assumption.

There exists $\epsilon > 0$ such that

$$p(x,x) > \epsilon.$$

Consequently P is analytic.

L^2-Gaffney estimates

Theorem (Coulhon, Grigor'yan, Zucca)

There exist $C, c > 0$ such that for all balls $B \subset \Gamma$ of radius r, all $j \in \mathbb{N}^*$ and all functions f supported in B,

$$\|P^{l-1} f\|_{L^2(C_j(B))} \leq C e^{-c\frac{4^j r^2}{l}} \|f\|_{L^2} \qquad \forall l \in \mathbb{N}^*.$$

Corollary (F.)

Let $M \in \mathbb{N}$. There exists $C_M, c_M > 0$ such that for all balls $B \subset \Gamma$ of radius r, all $j \in \mathbb{N}^*$ and all functions f supported in B,

$$\|(I - (I + r^2\Delta)^{-1})^M f\|_{L^2(C_j(B))} \leq C_M e^{-c_M 2^j} \|f\|_{L^2}$$

and

$$\|r\nabla(r^2\Delta)^M (I + r^2\Delta)^{-M-\frac{1}{2}} f\|_{L^2(C_j(B))} \leq C_M e^{-c_M 2^j} \|f\|_{L^2}$$

Hardy spaces via molecules

Let $M \in \mathbb{N}^*$ and $\epsilon > 0$. A function m is a (M,ϵ)-molecule if there exist a ball B of radius r and $b \in L^2(\Gamma)$ such that

1. $m = (I - (I + r^2\Delta)^{-1})^M b$,
2. $\|b\|_{L^2(C_j(B))} \leq \dfrac{2^{-j\epsilon}}{V(2^jB)}$ for all $j \in \mathbb{N}^*$.

The Hardy space $H^1_{mol,M,\epsilon}(\Gamma)$ is defined by

$$H^1_{mol,M,\epsilon}(\Gamma) = \{f = \sum_{i\in\mathbb{N}} \lambda_i m_i,\ (\lambda_i)_{i\in\mathbb{N}} \in \ell^1(\mathbb{N}),\\ (m_i)_{i\in\mathbb{N}} \text{ sequence of } (M,\epsilon)\text{-molecules}\}$$

The space $H^1_{mol,M,\epsilon}(\Gamma)$ is outfitted with the norm

$$\|f\|_{H^1_{mol,M,\epsilon}} = \inf\{\sum |\lambda_i|,\ f = \sum \lambda_i m_i\}$$

Hardy spaces via Lusin functionals

Define the functional L_β by

$$L_\beta f(x) := \left(\sum_{(y,l)\in\gamma(x)} \frac{(l+1)^{2\beta-1} m(y)}{V(x,\sqrt{l+1})} |\Delta^\beta P^l f(y)|^2 \right)^{\frac{1}{2}}$$

where $\gamma(x) := \left\{(y,l) \in \Gamma \times \mathbb{N},\, d(x,y)^2 \leq l\right\}$.

We introduce then $E^1_{quad,\beta}(\Gamma)$ as

$$E^1_{quad,\beta}(\Gamma) := \{f \in L^2(\Gamma),\, \|L_\beta f\|_{L^1} < +\infty\}$$

and is outfitted with the norm $\|f\|_{H^1_{quad,\beta}} := \|L_\beta f\|_{L^1}$.

Equivalence of Hardy spaces of functions

Theorem (F.)

For $M > \frac{d_0}{4}$, $\epsilon \in (0, +\infty)$ and $\beta > 0$, one has

$$E^1_{quad,\beta}(\Gamma) = H^1_{mol,M,\epsilon}(\Gamma) \cap L^2(\Gamma)$$

with equivalent norms. Consequently,

- $E^1_{quad,\beta}(\Gamma) \subset L^1(\Gamma)$,
- *the completion in $L^1(\Gamma)$ of $E^1_{quad,\beta}(\Gamma)$, called $H^1_{quad,\beta}(\Gamma)$, exists,*
- $H^1_{quad,\beta}(\Gamma) = H^1_{mol,M,\epsilon}(\Gamma)$.

These equivalent spaces will be written $H^1(\Gamma)$.

Remark : If we assume pointwise gaussian estimates of the Markov kernel $p(x, y)$, M can be choosen in $\mathbb{N}^*$.

Definition of spaces of 1-forms

- Define $T_x = \{(x,y) \in E\}$ for any $x \in \Gamma$.
- If $F_x : T_x \to \mathbb{R}$, then

$$|F_x|_{T_x} = \left(\frac{1}{2} \sum_{y \in \Gamma} p(x,y) |F_x(x,y)|^2 \right)^{\frac{1}{2}}.$$

- $T_\Gamma = \bigcup_{x \in \Gamma} T_x = E$.
- F is said to be in $\mathcal{F}(T_\Gamma)$ if it is a function on E such that $F(x,y) = -F(y,x)$.
- We say that $F \in L^p(T_\Gamma)$ if $F \in \mathcal{F}(T_\Gamma)$ and $x \in \Gamma \mapsto |F(x,.)|_{T_x} \in L^p(\Gamma)$.

Gradient and first properties

Define $d : \mathcal{F}(\Gamma) \to \mathcal{F}(T_\Gamma)$ and $d^* : \mathcal{F}(T_\Gamma) \to \mathcal{F}(\Gamma)$ by

$$df(x,y) = f(x) - f(y)$$

and

$$d^* F(x) = \sum_{y \in \Gamma} p(x,y) F(x,y).$$

Proposition

- $d^* d = \Delta$,
- $|df|_{T_x} = \nabla f(x)$.

Let $H^2(T_\Gamma) = \overline{d(L^2(\Gamma))} \subset L^2(T_\Gamma)$, then

- $d\Delta^{-\frac{1}{2}}$ is an isometry from $L^2(\Gamma)$ to $H^2(T_\Gamma)$,
- $\Delta^{-\frac{1}{2}} d^*$ is an isometry from $H^2(T_\Gamma)$ to $L^2(\Gamma)$,
- $d\Delta^{-1} d^* = Id_{H^2(T_\Gamma)}$.

Hardy spaces of 1-forms

Let $M \in \mathbb{N}$ and $\epsilon > 0$. A function m is a $(M + \frac{1}{2}, \epsilon)$-molecule if there exist a ball B of radius r and $b \in L^2(\Gamma)$ such that

1. $m = d\Delta^{-\frac{1}{2}}(I - (I + r^2\Delta)^{-1})^{M+\frac{1}{2}} b = r\nabla(r^2\Delta)^M(I + r^2\Delta)^{-M-\frac{1}{2}} b$
2. $\|b\|_{L^2(C_j(B))} \leq \dfrac{2^{-j\epsilon}}{V(2^j B)}$ for all $j \in \mathbb{N}^*$.

The Hardy space $H^1_{mol,M+\frac{1}{2},\epsilon}(T_\Gamma)$ is defined by

$$H^1_{mol,M,\epsilon}(T_\Gamma) = \{f = \sum_{i \in \mathbb{N}} \lambda_i m_i,\ (\lambda_i)_{i \in \mathbb{N}} \in \ell^1(\mathbb{N}),$$
$$(m_i)_{i \in \mathbb{N}} \text{ sequence of } (M + \frac{1}{2}, \epsilon)\text{-molecules}\Big\}$$

And we also introduce for $\beta > 0$,

$$E^1_{quad,\beta}(T_\Gamma) = \{F \in H^2(T_\Gamma),\ \|\Delta^{-\frac{1}{2}} d^* F\|_{H^1_{quad,\beta}}\}$$

Theorem (F.)

For $M+\frac{1}{2}>\frac{d_0}{4}$, $\epsilon\in(0,+\infty)$ and $\beta>0$, one has

$$E^1_{quad,\beta}(T_\Gamma)=H^1_{mol,M+\frac{1}{2},\epsilon}(T_\Gamma)\cap H^2(T_\Gamma)$$

with equivalent norms. Consequently,

- $E^1_{quad,\beta}(T_\Gamma)\subset L^1(T_\Gamma)$,
- *the completion in* $L^1(T_\Gamma)$ *of* $E^1_{quad,\beta}(T_\Gamma)$, *called* $H^1_{quad,\beta}(T_\Gamma)$, *exists,*
- $H^1_{quad,\beta}(T_\Gamma)=H^1_{mol,M+\frac{1}{2},\epsilon}(T_\Gamma)$.

These equivalent spaces will be named $H^1(T_\Gamma)$.

Remark : If we assume pointwise gaussian estimates of the Markov kernel $p(x,y)$, M can be choosen in $\mathbb{N}$.

Boundedness of Riesz transforms

Theorem (F.)

The Riesz transform $d\Delta^{-\frac{1}{2}}$ is $H^1(\Gamma)$-$H^1(T_\Gamma)$ bounded. Hence,

- *$d\Delta^{-\frac{1}{2}}$ is $H^1(\Gamma)$-$L^1(T_\Gamma)$ bounded and*
- *$\nabla\Delta^{-\frac{1}{2}}$ is $H^1(\Gamma)$-$L^1(\Gamma)$ bounded.*

Proof :

$$\|d\Delta^{-\frac{1}{2}}f\|_{H^1(T_\Gamma)} \simeq \|d\Delta^{-\frac{1}{2}}f\|_{H^1_{quad,1}(T_\Gamma)} = \|\Delta^{-\frac{1}{2}}d^*d\Delta^{-\frac{1}{2}}f\|_{H^1_{quad,1}(\Gamma)}$$
$$= \|f\|_{H^1_{quad,1}(\Gamma)} \simeq \|f\|_{H^1(\Gamma)}$$

Remark : Under pointwise gaussian estimates for the Markov kernel, we recover by interpolation the $L^p(\Gamma)$ boundedness of the Riesz transforms.

Thank you for your attention.

References

- P. Auscher, A. McIntosh and E. Russ. Hardy spaces of differential forms and Riesz transforms on Riemannian manifolds. *J. Geom. Anal.*, **18** (2008), pp. 192–248.
- F. Bernicot and J. Zhao. New abstract Hardy spaces. *J. Funct. Anal.*, **255** (2008), pp. 1761-1796.
- T. Coulhon, A. Grigor'yan and F. Zucca. The discrete integral maximum principle and its applications, *Tohoku Math. J.*, **57** (2005), pp. 559-587.