Sobolev algebras through semigroup methods

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Joint work with F. Bernicot (Nantes) and T. Coulhon (Canberra)

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The Euclidean case

On $\mathbb{R}^d$, the Bessel potential space

$$L^p_\alpha(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d); \Delta^{\alpha/2} f \in L^p(\mathbb{R}^d)\}, \qquad p \in (1,\infty),\ \alpha > 0,$$

is an algebra for the pointwise product when $\alpha p > d$ (Strichartz '67).

$$f, g \in L^p_\alpha \quad \Rightarrow \quad fg \in L^p_\alpha \ \ \textit{with} \ \ \|fg\|_{L^p_\alpha} \lesssim \|f\|_{L^p_\alpha} \|g\|_{L^p_\alpha}.$$

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Stronger results: The space

$$\dot{L}^p_\alpha(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \qquad p \in (1,\infty),\ \alpha > 0,$$

is an algebra for the pointwise product.
(Coifman, Meyer '78; Kato, Ponce '88; Gulisashvili, Kon '96).

$$f, g \in \dot{L}^p_\alpha \cap L^\infty \quad \Rightarrow \quad fg \in \dot{L}^p_\alpha \cap L^\infty \ \ \textit{with} \ \ \|fg\|_{\dot{L}^p_\alpha} \lesssim \|f\|_{\dot{L}^p_\alpha} \|g\|_\infty + \|f\|_\infty \|g\|_{\dot{L}^p_\alpha}.$$

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Coulhon, Russ, Tardivel-Nachef '01: Lie groups with polynomial volume growth, and Riemannian manifolds with non-negative Ricci curvature.
$\rightarrow$ in this setting, the Riesz transform is bounded for $p \in (1, \infty)$.

Badr, Bernicot, Russ '12: more general Riemannian manifolds.
$\rightarrow$ under Riesz transform bounds and Poincaré inequalities.

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Assumptions:

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- Heat kernel estimates: $(e^{-tL})_{t>0}$ has a kernel p_t satisfying

$$p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right), \quad \forall t > 0,\ \text{a.e. } x, y \in M.$$

Sobolev algebras for $\alpha = 1$

$\mathcal{C}_0(M)$: continuous functions on M which vanish at infinity.

Definition

For $p \in (1, \infty)$ and $\alpha > 0$, define $\dot{L}^p_\alpha(M)$ as the completion of

$$\{f \in \mathcal{C}_0(M);\ L^{\alpha/2} f \in L^p(M)\}$$

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$p \in (1,\infty)$. The Riesz transform is bounded on $L^p(M,\mu)$ if

$$(R_p) \qquad \| |\nabla f| \|_p \leq C \|\sqrt{L} f\|_p \qquad \forall f \in \mathcal{D},$$

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(R_p) and (RR_p) imply that $\dot{L}^p_1(M) \cap L^\infty(M)$ is an algebra.

Excursion

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For $p \in [1,\infty)$, one says that a L^p-Poincaré inequality holds if

$$\left(\fint_B \left|f - \fint_B f\,d\mu\right|^p d\mu\right)^{1/p} \le C r_B \left(\fint_B |\nabla f|^p\,d\mu\right)^{1/p} \qquad \forall f \in \mathcal{D},\ B \text{ ball in } M.$$

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Theorem (Auscher, Coulhon, Duong, Hofmann '04)

Suppose that a L^2-Poincaré inequality holds and, for some $p_0 \in (2, \infty)$,

$$(G_{p_0}) \qquad \sup_{t>0} \| \sqrt{t} |\nabla e^{-tL}| \|_{p_0 \to p_0} < \infty.$$

Then (R_p) holds for every $p \in [2, p_0)$.

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Improvement:

Theorem (Bernicot, Coulhon, F. '14)

Suppose that for some $p_0 \in (2, \infty)$, a L^{p_0}-Poincaré inequality holds and (G_{p_0}). Then (R_p) holds for every $p \in [2, p_0)$.

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The space $\dot{L}^p_\alpha(M) \cap L^\infty(M)$ *is an algebra for the pointwise product given that*

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CRT '01: gives (i) in a situation where (P_2) holds;
gives (iv).

BBR '12: gives (i) under extra condition (P_p);
gives (iii) under extra condition (P_2).

Methods I - Strichartz functional

Was used in Strichartz '67, CRT '01, BBR '12.

Characterisation of Sobolev spaces via quadratic functionals:

$$S_\alpha f(x) = \left(\int_0^\infty \fint_{B(x,r)} \left| f - \fint_{B(x,r)} f \, d\mu \right|^2 d\mu \frac{dr}{r^{1+2\alpha}} \right)^{1/2}.$$

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Proposition

Assume (G_{p_0}) for some $p_0 \in (2, \infty)$ and a L^{p_0}-Poincaré inequality. Then for every $\alpha \in (0,1)$, $p \in [2, p_0)$

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Conversely, *assume that for some $p > \nu$ and $\alpha \in (\frac{\nu}{p}, 1)$, we have*

$$\|f\|_{\dot{L}^p_\alpha} \simeq \|S_\alpha f\|_p.$$

Then (M, d, μ) admits a L^2-Poincaré inequality.

Approximation operators: $D \in \mathbb{N}$, $D > 2\nu$.

$$Q_t := (tL)^D e^{-tL},$$

$$P_t := \phi(tL), \quad \text{with } \phi(x) := \int_x^\infty s^D e^{-s} \frac{ds}{s}.$$

Note that $t\partial_t P_t = tL\phi'(tL) = -Q_t$, and $\lim_{t\to 0} P_t = c \cdot Id$ in $L^2(M)$.

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Product decomposition (modulo constants)

$$fg = \lim_{t\to 0}(P_t f \cdot P_t g) - \lim_{t\to\infty}(P_t f \cdot P_t g) = -\int_0^\infty \partial_t(P_t f \cdot P_t g) = \Pi_g(f) + \Pi_f(g).$$

Need to show

$$\|\Pi_g(f)\|_{\dot{L}^p_\alpha(M)} \leq C\|f\|_{\dot{L}^p_\alpha(M)}\|g\|_\infty.$$

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$$\begin{aligned} Tf := L^{\alpha/2}\Pi_g(L^{-\alpha/2}f) &= c\int_0^\infty\int_0^\infty Q_t L^{\alpha/2}(Q_s L^{-\alpha/2}f \cdot P_s g)\,\frac{ds}{s}\,\frac{dt}{t} \\ &= c\int_0^\infty\int_0^\infty K^\alpha(s,t)f\,\frac{ds}{s}\,\frac{dt}{t}, \end{aligned}$$

with $K^\alpha(s,t)f = Q_t L^{\alpha/2}(Q_s L^{-\alpha/2}f \cdot P_s g)$.

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Claim: *$T \in \mathcal{B}(L^p)$.*

(i) *$T \in \mathcal{B}(L^2)$ by form definition.*

Methods III - Extrapolation in L^p

Need to show
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(ii) ***For*** *$p \in (1,2)$**:** Extrapolation method [Duong, McIntosh '99, Blunck, Kunstmann '03, Auscher '07]: L^2 off-diagonal estimates*

$$\|TQ_r\|_{L^2(B_1)\to L^2(B_2)} \le C\left(1+\frac{d(B_1,B_2)^2}{r}\right)^{-N}.$$

Uses off-diagonal estimates for $K^\alpha(s,t)$.

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[Alternative method: SIO on tent spaces, see Auscher, Kriegler, Monniaux, Portal, '12.]

Theorem (Chain rule)

Suppose $F \in C^2(\mathbb{R})$, $F'' \in BUC(\mathbb{R})$, $F(0) = 0$. *Then*

$$f \in \dot{L}^p_\alpha \cap L^\infty \quad \Rightarrow \quad F(f) \in \dot{L}^p_\alpha \cap L^\infty,$$

given that

- $p \in (1, 2]$ *and* $\alpha \in (0, 1)$*;*
- $p \in (2, \nu)$ *and* $\alpha \in (0, 1)$*, and* (R_p) *holds.*

Remark: Need more regularity on F due to our methods (paraproducts instead of Strichartz functionals).

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Theorem (Paralinearisation)

Suppose $F \in C^3(\mathbb{R})$, $F(0) = 0$. Then

$$f \in \dot{L}^p_\alpha \cap L^\infty \quad \Rightarrow \quad F(f) - \Pi_{F'(f)}(f) \in \dot{L}^p_{\alpha+\rho} \cap L^\infty,$$

given that $p > \nu$, (G_p) *holds,* $\alpha \in (0, 1 - \frac{\nu}{p})$ *and* $\rho \in (0, \min\{1 - \frac{\nu}{p} - \alpha, \alpha - \frac{\nu}{p}\})$.

Compare with Bernicot, Sire '13.

References

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