# Sobolev algebras through semigroup methods

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# The Euclidean case

On $\mathbb{R}^d$, the Bessel potential space

$$L^p_\alpha(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d); \Delta^{\alpha/2} f \in L^p(\mathbb{R}^d)\}, \qquad p \in (1,\infty),\ \alpha > 0,$$

is an algebra for the pointwise product when $\alpha p > d$ (Strichartz '67).

$$f, g \in L^p_\alpha \quad \Rightarrow \quad fg \in L^p_\alpha \ \ \textit{with} \ \ \|fg\|_{L^p_\alpha} \lesssim \|f\|_{L^p_\alpha} \|g\|_{L^p_\alpha}.$$

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**Stronger results:** The space

$$\dot{L}^p_\alpha(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \qquad p \in (1,\infty),\ \alpha > 0,$$

is an algebra for the pointwise product.
(Coifman, Meyer '78; Kato, Ponce '88; Gulisashvili, Kon '96).

$$f, g \in \dot{L}^p_\alpha \cap L^\infty \quad \Rightarrow \quad fg \in \dot{L}^p_\alpha \cap L^\infty \ \ \textit{with} \ \ \|fg\|_{\dot{L}^p_\alpha} \lesssim \|f\|_{\dot{L}^p_\alpha} \|g\|_\infty + \|f\|_\infty \|g\|_{\dot{L}^p_\alpha}.$$

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**Coulhon, Russ, Tardivel-Nachef '01:** Lie groups with polynomial volume growth, and Riemannian manifolds with non-negative Ricci curvature.
$\rightarrow$ in this setting, the Riesz transform is bounded for $p \in (1, \infty)$.

**Badr, Bernicot, Russ '12:** more general Riemannian manifolds.
$\rightarrow$ under Riesz transform bounds and Poincaré inequalities.

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- $\mu$ Borel measure, finite on compact sets, strictly positive on any non-empty set, with doubling property

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- Heat kernel estimates: $(e^{-tL})_{t>0}$ has a kernel $p_t$ satisfying

$$p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right), \quad \forall t > 0,\ \text{a.e. } x,y \in M.$$

## Sobolev algebras for $\alpha = 1$

$\mathcal{C}_0(M)$: continuous functions on $M$ which vanish at infinity.

### Definition

For $p \in (1, \infty)$ and $\alpha > 0$, define $\dot{L}^p_\alpha(M)$ as the completion of

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$$(R_p) \qquad \||\nabla f|\|_p \leq C \|\sqrt{L} f\|_p \qquad \forall f \in \mathcal{D},$$

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*$(R_p)$ and $(RR_p)$ imply that $\dot{L}^p_1(M) \cap L^\infty(M)$ is an algebra.*

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For $p \in [1,\infty)$, one says that a $L^p$-Poincaré inequality holds if

$$\left( \fint_B \left| f - \fint_B f \, d\mu \right|^p d\mu \right)^{1/p} \le C r_B \left( \fint_B |\nabla f|^p \, d\mu \right)^{1/p} \qquad \forall f \in \mathcal{D}, \; B \text{ ball in } M.$$

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## Theorem (Auscher, Coulhon, Duong, Hofmann '04)

*Suppose that a $L^2$-Poincaré inequality holds and, for some $p_0 \in (2, \infty)$,*

$(G_{p_0})$
$$\sup_{t>0} \| \sqrt{t} |\nabla e^{-tL}| \|_{p_0 \to p_0} < \infty.$$

*Then $(R_p)$ holds for every $p \in [2, p_0)$.*

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Improvement:

**Theorem (Bernicot, Coulhon, F. '14)**

*Suppose that for some $p_0 \in (2,\infty)$, a $L^{p_0}$-Poincaré inequality holds and $(G_{p_0})$. Then $(R_p)$ holds for every $p \in [2, p_0)$.*

# Results - Sobolev algebras for $\alpha \in (0, 1)$

**Theorem (Bernicot, Coulhon, F. '14)**

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CRT '01: gives (i) in a situation where $(P_2)$ holds;
gives (iv).

BBR '12: gives (i) under extra condition $(P_p)$;
gives (iii) under extra condition $(P_2)$.

# Methods I - Strichartz functional

Was used in Strichartz '67, CRT '01, BBR '12.

Characterisation of Sobolev spaces via quadratic functionals:

$$S_\alpha f(x) = \left( \int_0^\infty \fint_{B(x,r)} \left| f - \fint_{B(x,r)} f \, d\mu \right|^2 d\mu \, \frac{dr}{r^{1+2\alpha}} \right)^{1/2}.$$

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## Proposition

*Assume $(G_{p_0})$ for some $p_0 \in (2,\infty)$ and a $L^{p_0}$-Poincaré inequality. Then for every $\alpha \in (0,1)$, $p \in [2, p_0)$*

$$\|f\|_{\dot{L}^p_\alpha} \simeq \|S_\alpha f\|_p.$$

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***Conversely,*** *assume that for some $p > \nu$ and $\alpha \in (\frac{\nu}{p}, 1)$, we have*

$$\|f\|_{\dot{L}^p_\alpha} \simeq \|S_\alpha f\|_p.$$

*Then $(M, d, \mu)$ admits a $L^2$-Poincaré inequality.*

# Methods II - Paraproducts

**Approximation operators:** $D \in \mathbb{N}$, $D > 2\nu$.

$$Q_t := (tL)^D e^{-tL},$$

$$P_t := \phi(tL), \quad \text{with } \phi(x) := \int_x^\infty s^D e^{-s} \, \frac{ds}{s}.$$

Note that $t\partial_t P_t = tL\phi'(tL) = -Q_t$, and $\lim_{t \to 0} P_t = c \cdot Id$ in $L^2(M)$.

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**Product decomposition** (modulo constants)

$$fg = \lim_{t\to 0}(P_t f \cdot P_t g) - \lim_{t\to\infty}(P_t f \cdot P_t g) = -\int_0^\infty \partial_t(P_t f \cdot P_t g) = \Pi_g(f) + \Pi_f(g).$$

*Need to show*

$$\|\Pi_g(f)\|_{\dot{L}^p_\alpha(M)} \leq C\|f\|_{\dot{L}^p_\alpha(M)}\|g\|_\infty.$$

# Methods III - Extrapolation in $L^p$

*Need to show*
$$\|\Pi_g(f)\|_{\dot{L}^p_\alpha(M)} \leq C\|f\|_{\dot{L}^p_\alpha(M)}\|g\|_\infty.$$

Write

$$\begin{aligned} Tf := L^{\alpha/2}\Pi_g(L^{-\alpha/2}f) &= c\int_0^\infty\int_0^\infty Q_tL^{\alpha/2}(Q_sL^{-\alpha/2}f\cdot P_sg)\,\frac{ds}{s}\,\frac{dt}{t} \\ &= c\int_0^\infty\int_0^\infty K^\alpha(s,t)f\,\frac{ds}{s}\,\frac{dt}{t}, \end{aligned}$$

with $K^\alpha(s,t)f = Q_tL^{\alpha/2}(Q_sL^{-\alpha/2}f\cdot P_sg)$.

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***Claim:*** $T \in \mathcal{B}(L^p)$.

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(i) *$T \in \mathcal{B}(L^2)$ by form definition.*

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(ii) ***For*** *$p \in (1,2)$**:** Extrapolation method [Duong, McIntosh '99, Blunck, Kunstmann '03, Auscher '07]: $L^2$ off-diagonal estimates*

$$\|TQ_r\|_{L^2(B_1)\to L^2(B_2)} \leq C\left(1+\frac{d(B_1,B_2)^2}{r}\right)^{-N}.$$

Uses off-diagonal estimates for $K^\alpha(s,t)$.

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# Methods III - Extrapolation in $L^p$

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$$\left(\fint_B |TP_{r^2}f\,d\mu|^p\right)^{1/p} \leq C\left(\inf_{x\in B}\mathcal{M}|S(f)|^2(x)\right)^{1/2}$$

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[Alternative method: SIO on tent spaces, see Auscher, Kriegler, Monniaux, Portal, '12. ]

**Theorem (Chain rule)**

*Suppose* $F \in C^2(\mathbb{R})$, $F'' \in BUC(\mathbb{R})$, $F(0) = 0$. *Then*

$$f \in \dot{L}^p_\alpha \cap L^\infty \quad \Rightarrow \quad F(f) \in \dot{L}^p_\alpha \cap L^\infty,$$

*given that*

- $p \in (1, 2]$ *and* $\alpha \in (0, 1)$;
- $p \in (2, \nu)$ *and* $\alpha \in (0, 1)$, *and* $(R_p)$ *holds.*

Remark: Need more regularity on $F$ due to our methods (paraproducts instead of Strichartz functionals).

## Results

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**Theorem (Paralinearisation)**

*Suppose $F \in C^3(\mathbb{R})$, $F(0) = 0$. Then*

$$f \in \dot{L}^p_\alpha \cap L^\infty \quad \Rightarrow \quad F(f) - \Pi_{F'(f)}(f) \in \dot{L}^p_{\alpha+\rho} \cap L^\infty,$$

*given that* $p > \nu$, $(G_p)$ *holds,* $\alpha \in (0, 1 - \frac{\nu}{p})$ *and* $\rho \in (0, \min\{1 - \frac{\nu}{p} - \alpha, \alpha - \frac{\nu}{p}\})$.

Compare with Bernicot, Sire '13.

# References


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