Hörmander functional calculus for non-selfadjoint operators

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Besançon – 02 October 2014

Introduction

Let f be a bounded measurable function on $(0,\infty)$ and $u(f)$ the operator on $L^p(\mathbb{R}^d)$ defined by

$$\widehat{u(f)g}(\xi) = f(|\xi|^2)\hat{g}(\xi).$$

For $p=2$, $u(f)$ is bounded. For $1<p\neq 2<\infty$, $u(f)$ is bounded if

$$\sup_{R>0}\int_{R/2}^{2R} |t^k f^{(k)}(t)|^2 \frac{dt}{t} < \infty \quad (k = 0,1,\dots,\lfloor \frac{d}{2}\rfloor + 1),$$

or (refined version) if for a $\phi \in C_c^\infty(0,\infty)$, $\phi\neq 0$,

$$\sup_{t>0} \|\phi f_t\|_{W_2^\alpha} =: \|f\|_{\mathcal{H}_2^\alpha} < \infty \text{ where } f_t(x) = f(tx) \text{ and } \alpha > \frac{d}{2}$$

[Hörmander 1960]. Besides $\mathcal{H}_2^\alpha$, consider also $\mathcal{H}_\infty^\alpha$ with $\|f\|_{\mathcal{H}_\infty^\alpha} = \sup_{t>0}\|\phi f_t\|_{W_\infty^\alpha}$. Then $\mathcal{H}_\infty^\alpha \subsetneq \mathcal{H}_2^\alpha$.

Introduction

Hörmander's theorem rewrites
$\|u(f)\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} = \|f(-\Delta)\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \leq C_p\|f\|_{\mathcal{H}^\alpha_2}$.
GOAL: Show a Hörmander functional calculus for abstract operators A in place of $-\Delta$.
SETTING: A 0-sectorial operator on some Banach space X, i.e. $(\exp(-zA))_{z\in\mathbb{C}_+}$ is an analytic semigroup bounded on subsectors $\Sigma_\theta = \{z \in \mathbb{C}\backslash\{0\} : \ |\arg(z)| < \theta\}$, $\theta < \frac{\pi}{2}$ of $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}$. Set $H^\infty(\Sigma_\omega) = \{f : \Sigma_\omega \to \mathbb{C} : \ f \text{ analytic and bounded}\}$. Define

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - A)^{-1}d\lambda$$

for $f \in H^\infty(\Sigma_\omega)$ for some $\omega \in (0,\pi)$ with polynomial decay at 0 and at ∞ and $\Gamma = \partial\Sigma_{\theta-\epsilon}$ the boundary of any subsector (Cauchy integral formula).

Introduction

One has $H^\infty(\Sigma_\omega) \subset \mathcal{H}^\alpha_\infty \subset \mathcal{H}^\alpha_2$ for any ω, α.
If $\|f(A)\|_{X\to X} \leq C\|f\|_{\infty,\Sigma_\omega}$ for all such f, then A is said to have a **(bounded)** $H^\infty(\Sigma_\omega)$ **calculus.**
If $\|f(A)\|_{X\to X} \leq C\|f\|_{\mathcal{H}^\alpha_q}$ for any $f \in H^\infty(\Sigma_\omega) \subset \mathcal{H}^\alpha_q$, then A is said to have a **(bounded)** $\mathcal{H}^\alpha_q$ **calculus,** $q = 2, \infty$.

Examples

1. If A is self-adjoint and positive on $X = L^2$, then A has an $H^\infty(\Sigma_\omega)$ calculus and an $\mathcal{H}_2^\alpha$ calculus for any $\omega \in (0,\pi)$, $\alpha > \frac{1}{2}$.
2. [Duong, Robinson 1996] Assume that (Ω, μ, ρ) is a space of homogeneous type, i.e. ρ distance, μ Borel measure, $\mu(B(x,r)) \leq Cr^d$. Assume that A is 0-sectorial on $L^2(\Omega)$ and $\exp(-zA)$ has an integral kernel $k_z(x,y)$ satisfying

$$(\widetilde{PE})_{\mathbb{C}_+} : \quad |k_z(x,y)| \leq \frac{C_{\arg(z)}}{\mu(B(x,|z|))} \frac{1}{|1+\rho(x,y)^2/z^2|^{\frac{d+1}{2}}} \quad (z \in \mathbb{C}_+).$$

Then if A has an $H^\infty(\Sigma_\omega)$ calculus on $L^2(\Omega)$, A has an $H^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega)$, $1 < p < \infty$.

Gaussian estimates

THEOREM: [Duong, Ouhabaz, Sikora 2002] Let (Ω, μ, ρ) be a space of homogeneous type with dimension d. Let A be self-adjoint positive on $L^2(\Omega)$ and assume that $\exp(-tA)$ has a kernel $k_t(x, y)$ with

$$(GE)_{\mathbb{R}_+} : \quad |k_t(x, y)| \leq C \frac{1}{\mu(B(x, t))} \exp(-c\rho(x, y)^2/t) \quad (t > 0).$$

Then A has a $\mathcal{H}^\alpha_\infty$ calculus on $L^p(\Omega)$ for $1 < p < \infty$ and $\alpha > \frac{d}{2}$.
REMARK: In many examples of $(GE)_{\mathbb{R}_+}$, A has the better $\mathcal{H}^\alpha_2$ calculus in place of $\mathcal{H}^\alpha_\infty$:
A left-invariant operator on homogeneous Lie group [Christ 1991/ Mauceri, Meda 1990], A elliptic differential operator of order 2 on compact Riemannian manifold [Seeger, Sogge 1989], A Schrödinger operator for certain potentials on compact connected manifolds [Duong, Ouhabaz, Sikora 2002], ...
But not in all.

$\mathcal{H}_2^\alpha$ calculus for Poisson estimates

MAIN THEOREM: Let (Ω, μ, ρ) be a space of homogeneous type such that $\mu(B(x,r)) \cong r^d$ at $r = 0$ and also $\mu(B(x,r,R))$ behaves as in the Euclidean case. Assume that A has an $H^\infty(\Sigma_\omega)$ calculus on $L^2(\Omega)$ for some $\omega \in (0,\pi)$ and $\exp(-zA)$ is an analytic semigroup on $L^2(\Omega)$ with kernel $k_z(x,y)$ satisfying

$$(PE)_{\mathbb{C}_+} : \quad |k_z(x,y)| \leq \frac{C}{\mu(B(x,|z|))} \frac{1}{|1+\rho(x,y)^2/z^2|^{\frac{d+1}{2}}} \quad (z \in \mathbb{C}_+).$$

Assume also that $\|\exp(-zA)\|_{L^2(\Omega)\to L^2(\Omega)} \leq C(\frac{\pi}{2} - |\arg(z)|)^{-\frac{d-1}{2}}$. Then A has an $\mathcal{H}_2^\alpha$ calculus on $L^p(\Omega)$, $1 < p < \infty$ for $\alpha > \frac{d}{2}$. Moreover, one has square function estimates

$$\left\| \left(\sum_{k=1}^n |f_k(A)x_k|^2 \right)^{\frac{1}{2}} \right\|_p \leq C \max_{k=1}^n \|f_k\|_{\mathcal{H}_2^\alpha} \cdot \left\| \left(\sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\|_p .$$

Remarks

REMARK 1: If A is self-adjoint on $L^2(\Omega)$ then the H^∞ calculus assumption and the $\|\exp(-zA)\|_{2\to 2}$ estimate are for free. But next example shows that the conditions of the theorem can be satisfied without self-adjointness. This should be regarded as the main novelty of MAIN THEOREM.

REMARK 2: Comparison of $(PE)_{\mathbb{C}_+}$ and $(GE)_{\mathbb{R}_+}$: Neither of these conditions is implied by the other. If $\rho(x,y) \to \infty$, then $\frac{1}{|1+\rho(x,y)^2/t^2|^{\frac{d+1}{2}}}$ obviously decays slower than $\exp(-c\rho(x,y)^2/t)$, so that $(PE)_{\mathbb{C}_+} \not\Longrightarrow (GE)_{\mathbb{R}_+}$. On the other hand, $(GE)_{\mathbb{R}_+}$ yields only real time estimates, so that $(GE)_{\mathbb{R}_+} \not\Longrightarrow (PE)_{\mathbb{C}_+}$.

REMARK 3: Variant of MAIN THEOREM: If one allows an additional factor $(\frac{\pi}{2} - |\arg(z)|)^{-\beta}$ for some $\beta > 0$ on the r.h.s. of $(PE)_{\mathbb{C}_+}$, then the weaker conclusion of an $\mathcal{H}_2^{\alpha+\beta}$ calculus holds.

An Example

Let $d \in \mathbb{N}$. Consider the operator L acting on functions $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{C}^{d+1}$ by

$$Lu = \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u, \; u = (u_1, \ldots, u_{d+1}),$$

with $\mu > 0$ and $2\mu + \lambda > 0$. L is called the **Lamé operator of elasticity.** Consider the Dirichlet problem on $\mathbb{R}^d \times \mathbb{R}_+$

$$\begin{cases} Lu & = 0 \text{ in } \mathbb{R}^d \times \mathbb{R}_+ \\ u|_{\mathbb{R}^d \times \{0\}} & = f \in L^p(\mathbb{R}^d; \mathbb{C}^{d+1}) \end{cases}.$$

Then $T_t f = u(\cdot, t)$ is a semigroup with some generator $-A$ acting on $L^p(\mathbb{R}^d; \mathbb{C}^{d+1})$. A satisfies the hypotheses of the MAIN THEOREM, in the variant with $\beta = 1$. Consequently, A has a $\mathcal{H}_2^{\frac{d}{2}+1+\epsilon}$ calculus for $\epsilon > 0$ and $1 < p < \infty$.

An Example

In the above example, replace $\mathbb{R}^d \times \mathbb{R}_+$ by some regular open subset Ω of $\mathbb{R}^{d+1}$. Then in [Kunstmann, Uhl 2012], a $\mathcal{H}_2^\alpha$ calculus for L on $L^p(\Omega)$ is established for certain α and $p \in (p_1, p_2)$. This is a **different** approach and neither result implies the other; indeed, L is self-adjoint, whereas A above is not.

$$(\exp(-tA)f)_\alpha = c_1 \int_{\mathbb{R}^d} \frac{t}{(|x-y|^2 + t^2)^{\frac{d+1}{2}}} f_\alpha(y)dy$$

$$+ c_2 \sum_{\gamma=1}^{d+1} \int_{\mathbb{R}^d} \frac{t(x-y,t)_\alpha (x-y,t)_\gamma}{(|x-y|^2+t^2)^{\frac{d+3}{2}}} f_\gamma(y)dy \quad (\alpha = 1, \dots, d+1).$$

[Martell, Mitrea, Mitrea, Mitrea 2012]

Further examples

1. If $A = (-\Delta)^{\frac{1}{2}}$ on $\Omega = \mathbb{R}^d$, the assumptions of the MAIN THEOREM are satisfied. It therefore yields square function estimates for spectral multipliers of $\mathcal{H}_2^\alpha$ functions.
2. Let Ω be a compact d-dimensional Riemannian manifold without boundary and A a classical, self-adjoint strongly elliptic pseudodifferential operator on Ω of order 1 such that $\inf\{\Re z : z \in \sigma(A)\} \geq 0$. Then by [Gimperlein, Grubb 2014], A satisfies the assumptions of MAIN THEOREM in the variant with $\beta = \frac{7}{2}d + 11$. If Ω satisfies the assumptions of MAIN THEOREM, then we deduce a $\mathcal{H}_2^\alpha$ calculus for A with $\alpha > \frac{d}{2} + \frac{7}{2}d + 11$. It is known from [Seeger, Sogge 1989] that the same holds with the better $\alpha > \frac{d}{2}$, so that MAIN THEOREM only yields new square function estimates for spectral multipliers.

Parts of the proof

1. Reduction to semigroup estimates: Let $f \in \mathcal{H}_2^\alpha$ with compact support in $(0, \infty)$.

$$
\begin{aligned}
f(A)x &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) \exp(itA)x dt \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t+i) \exp((it-1)A)x dt \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \underbrace{\hat{f}(t+i)(1+|t|)^\alpha}_{\in L^2(\mathbb{R})} \underbrace{(1+|t|)^{-\alpha} \exp((it-1)A)x}_{\text{to estimate}} dt
\end{aligned}
$$

2. Use the above cited Theorem [Duong, Robinson 1996] to pass from an $H^\infty(\Sigma_\omega)$ calculus of A on $L^2(\Omega)$ to an $H^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega)$.

Parts of the proof

3. The passage from functions with compact support to general $\mathcal{H}_2^\alpha$ functions as well as the square function estimate in MAIN THEOREM requires square function estimates of $\{\exp(-e^{i\theta}2^n sA) : n \in \mathbb{Z}\}$ depending on $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. For singular integral operators as in our case, [Duong, McIntosh 1999] gives sufficient conditions for the boundedness of $\exp(-zA)$ on $L^p(\Omega)$ in terms of

$$\sup_{y\in\Omega, t>0} \int_{\rho(x,y)\geq ct} |k_{z+t}(x,y) - k_z(x,y)| d\mu(x).$$

And in [Mo, Lu 2007], a vector valued version is established in terms of

$$B_z = \sup_{y\in\Omega, t>0} \int_{\rho(x,y)\geq ct} \sup_{n\in\mathbb{Z}} |k_{2^n z+t}(x,y) - k_{2^n z}(x,y)| d\mu(x),$$

this version yielding the required square function estimates. B_z in turn is estimated using the Poisson estimate $(PE)_{\mathbb{C}_+}$.

Thank you for your attention