Riesz Transforms and Poisson Semigroups

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Joint work with M. Junge and J. Parcet

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Classical Riesz transform

$f \in L_2(\mathbb{R}^n)$,

$$R = \nabla \triangle^{-\frac{1}{2}}.$$

with

$$\nabla = (\frac{\partial}{\partial x_1}, \cdots \frac{\partial}{\partial x_j}, \cdots \frac{\partial}{\partial x_n}), \triangle = -\frac{\partial^2}{\partial x^2} = \sum_j \frac{\partial^2}{\partial x_j^2}$$

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$$Rf = (R_1 f, R_2 f, \cdots R_n f)$$

with $R_i = \partial_i \Delta^{-\frac{1}{2}}$ the $i - th$ Riesz transform,

$$\widehat{R_i f}(\xi) = c \frac{\xi_i}{|\xi|} \widehat{f}(\xi), \xi \in \mathbb{R}^n.$$

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(Riesz; Stein/Meyer-Bakry/Pisier/Gundy/Varopoulos and many others)

$$\|Rf\|_p = \|(\sum_j |R_j f|^2)^{\frac{1}{2}}\|_p \simeq \|f\|_p, 1 < p < \infty,$$

$$\|\nabla f\|_p = \|(\sum_j |\partial_j f|^2)^{\frac{1}{2}}\|_p \simeq \|\Delta^{\frac{1}{2}} f\|_p, 1 < p < \infty,$$

Carré du Champ—P. A. Meyer's Gradient form

L: generator of a Markov semigroup on $L^p(M)$ (e.g. an elliptic differential operator)

$$\|\nabla_L f\|_p \simeq \|L^{\frac{1}{2}} f\|_p ?$$

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Observation: $L = \Delta = -\partial_x^2$ on $(\mathbb{R}, dx)$; Chain rule:

$$2\nabla f_1 \cdot \nabla f_2 = -\Delta(f_1 f_2) + (\Delta f_1) f_2 + f_1 \Delta f_2$$

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$$2\Gamma(f_1, f_2) \quad = \quad -L(\bar{f}_1 f_2) + L(\bar{f}_1) f_2 + \bar{f}_1 L(f_2);$$

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$$\begin{aligned} 2\Gamma(f_1, f_2) &= -L(\bar{f}_1 f_2) + L(\bar{f}_1) f_2 + \bar{f}_1 L(f_2); \\ \Gamma(f, f) &< -- > |\nabla_L f|^2 \end{aligned}$$

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Questions

$$\|\Gamma(f, f)\|_{p/2}^2 \simeq \|L^{\frac{1}{2}} f\|_p?$$

Semiclassical Riesz transform —Markov Semigroups of Operators

$(\mathcal{M}, \mu)$: Sigma finite measure space,
$(S_t)_{t\geq 0}$: a semigroup of operators on $L_\infty(\mathcal{M})$
We say $(S_t)_t$ is Markov, if

- S_t are contractions on $L_\infty(\mathcal{M}, \mu)$.
- S_t are symmetric i.e. $\langle S_t f, g\rangle = \langle f, S_t g\rangle$ for $f, g \in L^1(\mathcal{M}) \cap L_\infty(\mathcal{M})$.
- $S_t(1) = 1$
- $S_t(f) \to f$ in the w^* topology for $f \in L_\infty(\mathcal{M})$.

Infinitesimal generator: $L = -\frac{\partial S_t}{\partial t}|_{t=0}; S_t = e^{-tL}$.
More general case: $L_\infty(\mathcal{M})$ replaced by semi finite von Neuman algebras.

Abstract theories by E. Stein, Cowling, Mcintoch...., Junge/Xu, Le Merdy-Junge-Xu.

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- (M, dx): complete Riemannian manifold.
  $Lf(x) = \sum_{i,j} a_{ij}(x)\partial_i\partial_j f + \sum_i g_i(x)\partial_i f$.
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  $\Gamma(\lambda_g, \lambda_h) = (\sum_{i=1}^{j} k_i k'_i) \lambda_{g^{-1}h}$ with
  j the first integer that $k_j \neq k'_j$.

L_p Fourier multipliers on Discrete Groups

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Question Is the fourier multiplier $\lambda_g \mapsto \frac{k_1}{\|g\|}\lambda_g$, for $g = a^{k_1} b^{k_2} a^{k_3} b^{k_4} \ldots . a^{k_n} \in \mathbb{F}_2$, (completely) bounded on $L^p(\hat{\mathbb{F}}_2)(1 < p < \infty)$?

Semiclassical Riesz transform—P.A. Meyer's question

Theorem (D.Bakry for diffusion S_t 1986;) Assume L generates a diffusion Markov semigroup (on commutative L_p spaces) satisfying $\Gamma_2 \geq 0$, then

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Junge; Junge/M 2010 noncommutative extension for $p > 2$.

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Let $L = \Delta^{\frac{1}{2}}$. For $p \leq \frac{2n}{n+1}$, and any Schwarz function f on $\mathbb{R}^n$,

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Oberservation $e^{i\xi\cdot} \mapsto e^{-t|\xi|} e^{i\xi\cdot}$ is a Markov semigroup
$\to \phi(\xi) = |\xi|$ is a conditionally negative function on $\mathbb{R}^n$
$\to$ there is an embedding $b : \mathbb{R}^n \mapsto H$ such that
$|\xi - \eta| = \|b(\xi) - b(\eta)\|^2$.

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= \; & \frac{|\xi - \eta|^{2\varepsilon} - |\xi|^{2\varepsilon} - |\eta|^{2\varepsilon}}{2} e^{i(-\xi+\eta)\cdot y} \\
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v_k : orthonormal basis of H.

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$L_p(\mathbb{R}^n \rtimes L_2(\Omega))$ is the L_p space of functions on $\Omega \times \mathbb{R}^n$ with a noncommutative product rule

$$(e^{i\langle \xi, \cdot \rangle} \cdot \gamma_k)(e^{i\langle \eta, \cdot \rangle} \cdot \gamma_j) = e^{i\langle \xi+\eta, \cdot \rangle} \cdot (\alpha_\eta(\gamma_k)\gamma_j).$$

L_p bound of Riesz transforms via cocycles

Theorem (Junge/M/Parcet, 2014) For any discrete group G with a cocycle (b, H, α), we have

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Pisier's rotation formula.

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Khintchine inequality for crossed products.
$\widetilde{b}_j$ is a twist of b_j coming from the Khintchine inequality for crossed products.

Riesz transform on free groups

$G = \mathbb{F}_2$: the free group of two generators a, b.
$\|g\|^2 = k_1^2 + k_2^2 + ...k_n^2$ for $g = a^{k_1} b^{k_2} a^{k_3} b^{k_4} ....a^{k_n}$.
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Question Is the following fourier multiplier (completely) bounded on $L^p(\hat{\mathbb{F}}_2)$?

$$\lambda_g \mapsto (\frac{k_1}{\|g\|}\lambda_g, ... \frac{k_i}{\|g\|}\lambda_g, ... \frac{k_n}{\|g\|}\lambda_g).$$

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For $g = a^{k_1}b^{k_2}a^{k_3}b^{k_4}....a^{k_n}$, $h = a^{k'_1}b^{k'_2}a^{k'_3}b^{k'_4}....a^{k'_n}$

$$\Gamma(\lambda_g, \lambda_h) = \frac{1}{2}(\|g\|^2 + \|h\|^2 - \|g^{-1}h\|^2)\lambda_{g^{-1}h} = \sum_{i=1}^{j} k_i k'_i \lambda_{g^{-1}h}$$

with j the first integer such that $k_j \neq k'_j$.

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$\Gamma(f_1, f_2) = \nabla_L f_1 \cdot \nabla_L f_2$ with $\nabla_L = (...\partial_h, ...)$ $h \in \mathbb{F}_2$

$$\partial_a(\lambda_g) = k_1, \partial_{a^{k_1} b}(\lambda_g) = k_2, \partial_{a^{k_1} b^{k_2} a}(\lambda_g) = k_3, ..$$

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Our main theorem says that $\nabla_L L^{-\frac{1}{2}}$ is bounded on $L^p(\widehat{\mathbb{F}}_2)$ in the sense that

$$\|(\sum_h |\partial_h L^{-\frac{1}{2}} f|^2)^{\frac{1}{2}}\|_{L_p(\widehat{\mathbb{F}}_2)} + \|(\sum_h |\partial_h L^{-\frac{1}{2}} f^*|^2)^{\frac{1}{2}}\|_{L_p(\widehat{\mathbb{F}}_2)} \simeq \|f\|_{L_p(\widehat{\mathbb{F}}_2)}$$

for $2 \leq p < \infty$, f being linear combinations of λ_g.

Classical Hörmander-Mihlin multipliers are averages of Riesz transforms

- Let $G = \mathbb{R}^n$.
  $H = L_2(\mathbb{R}^n, \frac{dx}{|x|^{n+2\varepsilon}})$. $b : \xi \in \mathbb{R}^n \to e^{i\langle \xi, \cdot \rangle} - 1 \in H$.
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- For $v \in H$, let

$$R_v : e^{i\langle \xi, x\rangle} \to \frac{\langle b(g), v\rangle}{\|b(\xi)\|} e^{i\langle \xi, x\rangle}.$$

Given $T_m : \int \hat{f}(\xi) e^{i\langle \xi, x\rangle} \to \int \hat{f}(\xi) m(\xi) e^{i\langle \xi, x\rangle} d\xi$, then

$$T_m = R_v$$

with

$$v(x) = |x|^{n+2\varepsilon} \widehat{m(\cdot)|\cdot|}^{\varepsilon}.$$

Hörmander-Mihlin multipliers on a branch of free groups

Given a branch B of $G = \mathbb{F}_\infty$, let

$$L_p(\widehat{\mathbf{B}}) = \{f = \sum_{g \in B} a_g \lambda_g ; \|f\|_{L^p(\widehat{\mathbb{G}})} = (\tau |f|^p)^{\frac{1}{p}} < \infty\}.$$

For $m : \mathbb{Z}_+ \to \mathbb{C}$, let

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Theorem (Junge/M/Parcet, 2014)
Suppose $m : \mathbb{Z}_+ \to \mathbb{C}$ satisfies

$$\sup_{j \geq 1} |m(j)| + j|m(j) - m(j-1)| < c$$

then

$$\|T_m f\|_{L^p(\widehat{\mathbb{G}})} \lesssim_{c(p)} c\|f\|_{L^p(\widehat{\mathbb{G}})}$$

for any $f \in L_p(\widehat{\mathbf{B}})$.

Littlewood-Paley estimates.

Theorem (Junge/M /Parcet, 2014) Consider a standard Littlewood-Paley partition of unity $(\varphi_j)_{j\geq 1}$ in $\mathbb{R}_+$. Let $\Lambda_j : \lambda(g) \mapsto \sqrt{\varphi_j(|g|)}\lambda(g)$ denote the corresponding radial multipliers in $\mathcal{L}(\mathbb{F}_\infty)$. Then, the following estimates hold for $f \in L_p(\widehat{\mathbf{B}})$ and $1 < p < 2$

$$\inf_{\Lambda_j f = a_j + b_j} \Big\| \Big(\sum_{j\geq 1} a_j^* a_j + \widetilde{b}_j \widetilde{b}_j^* \Big)^{\frac{1}{2}} \Big\|_{L^p(\widehat{\mathbb{G}})} \lesssim_{c(p)} \|f\|_{L^p(\widehat{\mathbb{G}})},$$

$$\|f\|_{L^p(\widehat{\mathbb{G}})} \lesssim_{c(p)} \inf_{\Lambda_j f = a_j + b_j} \Big\| \Big(\sum_{j\geq 1} a_j^* a_j + b_j b_j^* \Big)^{\frac{1}{2}} \Big\|_{L^p(\widehat{\mathbb{G}})}.$$