

Riesz Transforms and Poisson Semigroups

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Joint work with M. Junge and J. Parcet

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Classical Riesz transform

$$f \in L_2(\mathbb{R}^n),$$

$$R = \nabla \Delta^{-\frac{1}{2}}.$$

with

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_j}, \dots, \frac{\partial}{\partial x_n} \right), \quad \Delta = -\frac{\partial^2}{\partial x^2} = \sum_j \frac{\partial^2}{\partial x_j^2}$$

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$$Rf = (R_1f, R_2f, \dots, R_nf)$$

with $R_i = \partial_i \Delta^{-\frac{1}{2}}$ the i -th Riesz transform,

$$\widehat{R_i f}(\xi) = c \frac{\xi_i}{|\xi|} \widehat{f}(\xi), \xi \in \mathbb{R}^n.$$

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(Riesz; Stein/Meyer-Bakry/Pisier/Gundy/Varopoulos and many others)

$$\|Rf\|_p = \left\| \left(\sum_j |R_j f|^2 \right)^{\frac{1}{2}} \right\|_p \simeq \|f\|_p, 1 < p < \infty,$$

$$\|\nabla f\|_p = \left\| \left(\sum_i |\partial_i f|^2 \right)^{\frac{1}{2}} \right\|_p \simeq \|\Delta^{\frac{1}{2}} f\|_p, 1 < p < \infty,$$

Carré du Champ—P. A. Meyer's Gradient form

L : generator of a Markov semigroup on $L^p(M)$ (e.g. an elliptic differential operator)

$$\|\nabla_L f\|_p \simeq \|L^{\frac{1}{2}} f\|_p?$$

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Observation: $L = \Delta = -\partial_x^2$ on (\mathbb{R}, dx) ; Chain rule:

$$2\nabla f_1 \cdot \nabla f_2 = -\Delta(f_1 f_2) + (\Delta f_1) f_2 + f_1 \Delta f_2$$

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$$2\Gamma(f_1, f_2) = -L(\bar{f}_1 f_2) + L(\bar{f}_1) f_2 + \bar{f}_1 L(f_2);$$

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Questions

$$\|\Gamma(f, f)\|_{p/2}^2 \simeq \|L^{\frac{1}{2}} f\|_p^2?$$

Semiclassical Riesz transform — Markov Semigroups of Operators

(\mathcal{M}, μ) : Sigma finite measure space,

$(S_t)_{t \geq 0}$: a semigroup of operators on $L_\infty(\mathcal{M})$

We say $(S_t)_t$ is **Markov**, if

- ▶ S_t are contractions on $L_\infty(\mathcal{M}, \mu)$.
- ▶ S_t are symmetric i.e. $\langle S_t f, g \rangle = \langle f, S_t g \rangle$ for $f, g \in L^1(\mathcal{M}) \cap L_\infty(\mathcal{M})$.
- ▶ $S_t(1) = 1$
- ▶ $S_t(f) \rightarrow f$ in the w^* topology for $f \in L_\infty(\mathcal{M})$.

Infinitesimal generator: $L = -\frac{\partial S_t}{\partial t} \Big|_{t=0}$; $S_t = e^{-tL}$.

More general case: $L_\infty(\mathcal{M})$ replaced by semi finite von Neuman algebras.

Abstract theories by E. Stein, Cowling, McIntoch..., Junge/Xu, Le Merdy-Junge-Xu.

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- ▶ (M, dx) : complete Riemannian manifold.

$$Lf(x) = \sum_{i,j} a_{ij}(x) \partial_i \partial_j f + \sum_i g_i(x) \partial_i f.$$

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j the first integer that $k_j \neq k'_j$.

L_p Fourier multipliers on Discrete Groups

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Example: $G = \mathbb{Z}, \lambda_k = e^{ik\theta}, L_\infty(\hat{G}) = L_\infty(\mathbb{T})$.

τ : For $f = \sum_g f_g \lambda_g$,

$$\tau f = f_e.$$

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Question Is the Fourier multiplier $\lambda_g \mapsto \frac{k_1}{\|g\|} \lambda_g$, for $g = a^{k_1} b^{k_2} a^{k_3} b^{k_4} \dots a^{k_n} \in \mathbb{F}_2$, (completely) bounded on $L^p(\hat{\mathbb{F}}_2) (1 < p < \infty)$?

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Theorem (D.Bakry for diffusion S_t 1986;) Assume L generates a diffusion Markov semigroup (on commutative L_p spaces) satisfying $\Gamma_2 \geq 0$, then

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Junge; Junge/M 2010 noncommutative extension for $p > 2$.

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Let $L = \Delta^{\frac{1}{2}}$. For $p \leq \frac{2n}{n+1}$, and any Schwarz function f on \mathbb{R}^n ,

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Observation $e^{i\xi \cdot} \mapsto e^{-t|\xi|} e^{i\xi \cdot}$ is a Markov semigroup

$\rightarrow \phi(\xi) = |\xi|$ is a conditionally negative function on \mathbb{R}^n

\rightarrow there is an embedding $b : \mathbb{R}^n \mapsto H$ such that

$$|\xi - \eta| = \|b(\xi) - b(\eta)\|^2.$$

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$L : \lambda_g \mapsto \|b(g)\|^2 \lambda_g$ generates a Markov semigroup on $L_\infty(\widehat{G})$.

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i.e. $b : G \mapsto H, \alpha : G \mapsto \text{Aut}(H), \alpha_g b(g^{-1}h) = b(h) - b(g)$

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Cocycle for Fractional power of Laplacian

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v_k : orthonormal basis of H .

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$L_p(\mathbb{R}^n \rtimes L_2(\Omega))$ is the L_p space of functions on $\Omega \times \mathbb{R}^n$ with a noncommutative product rule

$$(e^{i\langle \xi, \cdot \rangle} \cdot \gamma_k)(e^{i\langle \eta, \cdot \rangle} \cdot \gamma_j) = e^{i\langle \xi + \eta, \cdot \rangle} \cdot (\alpha_\eta(\gamma_k) \gamma_j).$$

L_p bound of Riesz transforms via cocycles

Theorem (Junge/M/Parcet, 2014) For any discrete group G with a cocycle (b, H, α) , we have

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Pisier's rotation formula.

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Khintchine inequality for crossed products.

\tilde{b}_j is a twist of b_j coming from the Khintchine inequality for crossed products.

Riesz transform on free groups

$G = \mathbb{F}_2$: the free group of two generators a, b .

$\|g\|^2 = k_1^2 + k_2^2 + \dots k_n^2$ for $g = a^{k_1} b^{k_2} a^{k_3} b^{k_4} \dots a^{k_n}$.

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Question Is the following Fourier multiplier (completely) bounded on $L^p(\hat{\mathbb{F}}_2)$?

$$\lambda_g \mapsto \left(\frac{k_1}{\|g\|} \lambda_g, \dots, \frac{k_i}{\|g\|} \lambda_g, \dots, \frac{k_n}{\|g\|} \lambda_g \right).$$

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For $g = a^{k_1} b^{k_2} a^{k_3} b^{k_4} \dots a^{k_n}$, $h = a^{k'_1} b^{k'_2} a^{k'_3} b^{k'_4} \dots a^{k'_n}$

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Our main theorem says that $\nabla_L L^{-\frac{1}{2}}$ is bounded on $L^p(\widehat{\mathbb{F}}_2)$ in the sense that

$$\left\| \left(\sum_h |\partial_h L^{-\frac{1}{2}} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\widehat{\mathbb{F}}_2)} + \left\| \left(\sum_h |\partial_h L^{-\frac{1}{2}} f^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\widehat{\mathbb{F}}_2)} \simeq \|f\|_{L^p(\widehat{\mathbb{F}}_2)}$$

for $2 \leq p < \infty$, f being linear combinations of λ_g .

Classical Hörmander-Mihlin multipliers are averages of Riesz transforms

► Let $G = \mathbb{R}^n$.

$$H = L_2(\mathbb{R}^n, \frac{dx}{|x|^{n+2\varepsilon}}). \quad b : \xi \in \mathbb{R}^n \rightarrow e^{i\langle \xi, \cdot \rangle} - 1 \in H.$$

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- ▶ For $v \in H$, let

$$R_v : e^{i\langle \xi, x \rangle} \rightarrow \frac{\langle b(\xi), v \rangle}{\|b(\xi)\|} e^{i\langle \xi, x \rangle}.$$

Given $T_m : \int \hat{f}(\xi) e^{i\langle \xi, x \rangle} \rightarrow \int \hat{f}(\xi) m(\xi) e^{i\langle \xi, x \rangle} d\xi$, then

$$T_m = R_v$$

with

$$v(x) = |x|^{n+2\varepsilon} \widehat{m(\cdot)} \cdot |^\varepsilon.$$

Hörmander-Mihlin multipliers on a branch of free groups

Given a branch B of $G = \mathbb{F}_\infty$, let

$$L_p(\widehat{\mathbf{B}}) = \left\{ f = \sum_{g \in B} a_g \lambda_g; \|f\|_{L^p(\widehat{\mathbb{G}})} = (\tau |f|^p)^{\frac{1}{p}} < \infty \right\}.$$

For $m : \mathbb{Z}_+ \rightarrow \mathbb{C}$, let

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Theorem (Junge/M/Parcet, 2014)

Suppose $m : \mathbb{Z}_+ \rightarrow \mathbb{C}$ satisfies

$$\sup_{j \geq 1} |m(j)| + j|m(j) - m(j-1)| < c$$

then

$$\|T_m f\|_{L^p(\widehat{\mathbb{G}})} \lesssim_{c(p)} c \|f\|_{L^p(\widehat{\mathbb{G}})}$$

for any $f \in L_p(\widehat{\mathbf{B}})$.

Littlewood-Paley estimates.

Theorem (Junge/M /Parcet, 2014) Consider a standard Littlewood-Paley partition of unity $(\varphi_j)_{j \geq 1}$ in \mathbb{R}_+ . Let $\Lambda_j : \lambda(g) \mapsto \sqrt{\varphi_j(|g|)}\lambda(g)$ denote the corresponding radial multipliers in $\mathcal{L}(\mathbb{F}_\infty)$. Then, the following estimates hold for $f \in L_p(\widehat{\mathbf{B}})$ and $1 < p < 2$

$$\inf_{\Lambda_j f = a_j + b_j} \left\| \left(\sum_{j \geq 1} a_j^* a_j + \tilde{b}_j \tilde{b}_j^* \right)^{\frac{1}{2}} \right\|_{L^p(\widehat{\mathbb{G}})} \lesssim_{c(p)} \|f\|_{L^p(\widehat{\mathbb{G}})},$$

$$\|f\|_{L^p(\widehat{\mathbb{G}})} \lesssim_{c(p)} \inf_{\Lambda_j f = a_j + b_j} \left\| \left(\sum_{j \geq 1} a_j^* a_j + b_j b_j^* \right)^{\frac{1}{2}} \right\|_{L^p(\widehat{\mathbb{G}})}.$$