Spectral multipliers and restriction estimates via the derivatives of the corresponding semigroup

El Maati Ouhabaz, Univ. Bordeaux

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Our aim: suppose minimal regularity conditions on F: spectral multiplier results.

Theorem (Duong-Ou-Sikora, 2002)

Suppose that (X, ρ, μ) is a (open subset of) space of homogeneous type with homogeneous "dimension" d. Suppose that the heat kernel $p(t, x, y)$ of L satisfies the Gaussian upper bound

$$|p(t, x, y)| \le \frac{Ce^{-c\frac{\rho^2(x,y)}{t}}}{v(x, \sqrt{t})}, \ t > 0, x, y \in X.$$

If $\sup_{t>0} \|F(t\cdot)\varphi(\cdot)\|_{W^{\beta,\infty}} < \infty$ for some non-trivial $\varphi \in C_c^\infty(0, \infty)$ and some $\beta > d/2$, then $F(L)$ is of weak type $(1, 1)$ and extends to a bounded operator on L^p for all $p \in (1, \infty)$.

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Theorem (Hörmander)

If $\sup_{t>0} \|F(t\cdot)\varphi(\cdot)\|_{W^{\beta,2}} < \infty$ for some non-trivial $\varphi \in C_c^\infty(0,\infty)$ and some $\beta > d/2$, then the Fourier multiplier $F(-\Delta)$ is of weak type $(1,1)$ and extends to a bounded operator on $L^p(\mathbb{R}^d)$ for all $p \in (1,\infty)$.

Extension to Lie group settings by Christ, Mauceri-Meda, Alexopoulos....

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$$(\mathrm{ST}^{q}_{p,s}) \qquad \big\|F(\sqrt{L})P_{B(x,r)}\big\|_{p\to s} \le CV(x,r)^{\frac{1}{s}-\frac{1}{p}}\,(Rr)^{d(\frac{1}{p}-\frac{1}{s})}\big\|F(R\cdot)\big\|_{q}$$

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On $\mathbb{R}^d$, the restriction of the Fourier transform to the sphere S^{d-1},

$$R_\lambda f(\omega) := \hat{f}(\lambda\omega), \ \ \omega \in \mathbb{S}^{n-1}, \lambda > 0$$

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For this reason, we call $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ a *Stein-Tomas restriction type condition* and (R_p) the *Stein-Tomas* $(p, 2)$ *restriction condition.*

In order to state some of our results for F with less regularity, we recall that L satisfies the finite speed propagation property if the kernel of $\cos(t\sqrt{L})$ satisfies:

$$\text{(FS)} \qquad \operatorname{Supp} K_{\cos(t\sqrt{L})} \subseteq \{(x, y) \in X \times X : \rho(x, y) \leq t\} \quad \forall t > 0\,.$$

Property (FS) holds for most of second order self-adjoint operators and is equivalent to Davies-Gaffney estimates.

Theorem (P.Chen, E.M. Ou, A. Sikora, L. Yan, 2012)

Suppose that L satisfies the finite speed propagation property and the restriction estimate $(\mathrm{ST}^{q}_{p,s})$ *for some* p, s, q *such that* $1 \leq p < s \leq \infty$ *and* $1 \leq q \leq \infty$.

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(i) **Compactly supported multipliers:** *Let F be an even function such that $\mathrm{Supp} F \subseteq [-1, 1]$ and $F \in W^{\beta,q}$ for some $\beta > d(1/p - 1/s)$. Then $F(\sqrt{L})$ is bounded on $L^p(X)$, and*

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(ii) **General multipliers:** *Suppose $s = 2$ and F satisfies*

$$\sup_{t>0} \|F(t\cdot)\varphi(\cdot)\|_{W^{\beta,q}} < \infty$$

for some $\beta > \max\{d(1/p - 1/2), 1/q\}$ and some non-trivial function $\varphi \in C_c^\infty(0, \infty)$. Then $F(\sqrt{L})$ is bounded on $L^r(X)$ for all $p < r < p'$.

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For operators not satisfying (FS), a similar result is proved recently by Kunstmann and Uhl.

As a consequence (under $(\mathrm{ST}^{q}_{p,2})$):
for all $\delta > \max\{d(1/p - 1/2) - 1/q, 0\} =: \delta_q(p)$ the Bochner-Riesz mean of order δ, $S_R^\delta(L)$ with

$$S_R^\delta(\lambda) = \begin{cases} \left(1 - \frac{\lambda}{R^2}\right)^\delta & \text{for } \lambda \le R^2 \\ 0 & \text{for } \lambda > R^2. \end{cases}$$

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is bounded on L^p, uniformly in $R > 0$. The endpoint result is also true

Theorem

Assume that L satisfies the finite speed propagation property and the restriction condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ for some p, q satisfying $1 \leq p < 2$ and $1 \leq q \leq \infty$. Then the Bochner Riesz mean $S_R^{\delta_q(p)}(L)$ is of weak-type (p, p) uniformly in R.

In the Euclidean case, Christ and Sogge proved weak-type $(1, 1)$ for $S_R^{\delta_2(1)}(-\Delta)$. Weak-type (p, p) estimates of $S_R^{\delta_2(p)}(-\Delta)$ are proved by Christ when $p < \frac{2d+2}{d+3}$. The endpoint estimates for $p = \frac{2d+2}{d+3}$ are proved by Tao. Bochner-Riesz summability for $-\Delta$ on $\mathbb{R}^d$ holds $p \leq \frac{2d+4}{d+4}$ (due to S. Lee 2004) and improved recently by Bourgain-Guth '2011.

Links to dispersive or Strichartz estimates

Strichartz estimates for the Schrödinger equation associated to L:

$$\partial_t u + iLu = 0, \;\; u(0) = f \in L^2$$

read as follows:

$$\int_{\mathbb{R}} \|e^{itL} f\|^2_{\frac{2d}{d-2}} \, dt \leq C\|f\|_2^2, \;\; f \in L^2.$$

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Theorem

(i) *Suppose that L satisfies the Strichartz estimate and the classical smoothing property*

$$\|\exp(-tL)\|_{p \to \frac{2d}{d+2}} \le K t^{-\frac{d}{2}(\frac{1}{p} - \frac{d+2}{2d})},$$

for all $p \in [1, \frac{2d}{d+2}]$. *Then the restriction estimate* (R_p) *is satisfied.*

(ii) *Fix* $p \in [1, \frac{2d}{d+2}]$. *Suppose that* $V(x, r) \sim r^d$. *Assume that L satisfies the finite speed propagation property together with Strichartz and smoothing estimates as in* (i). *Then the previous sharp spectral multiplier results hold with regularity* $W^{\beta,2}$ *for* $\beta > d(1/p - 1/2)$.

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$$\|e^{itL}\|_{1\to\infty} \le C|t|^{-d/2}, \quad t \in \mathbb{R},\ t \neq 0$$

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- Is the limitation in p, i.e. $p \le \frac{2d}{d+2}$ optimal in our abstract setting ? could one push this to $\frac{2d+2}{d+3}$?

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Theorem (Bernicot-Ou 2013)

Let d be a positive constant and fix $p \in [1,2)$. The following assertions are equivalent.
1) The restriction estimate (R_p) holds for every $\lambda > 0$;
2) There exists a positive constant C such that

$$\|L^N e^{-tL}\|_{L^p \to L^{p'}} \leq C(N-1)! N^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})} t^{-N-\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})},$$

for all $t > 0$ and all $N \in \mathbb{N}$.

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The main ingredient for the proof is the following formula for the functional calculus:

$$\lim_{N\to\infty} \frac{1}{(N-1)!} \int_0^\infty \phi(s^{-1}) \langle ((N-1)sL)^N e^{-s(N-1)L} f, g\rangle \frac{ds}{s} = \langle \phi(L) f, g\rangle.$$

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2) There exists a positive constant C such that

$$\|L^N e^{-tL}\|_{L^p \to L^{p'}} \leq C(N-1)! N^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})} t^{-N-\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})},$$

for all $t > 0$ and all $N \in \mathbb{N}$.

The main ingredient for the proof is the following formula for the functional calculus:

$$\lim_{N\to\infty} \frac{1}{(N-1)!} \int_0^\infty \phi(s^{-1}) \langle ((N-1)sL)^N e^{-s(N-1)L} f, g \rangle \frac{ds}{s} = \langle \phi(L) f, g \rangle.$$

Hence

$$dE_L(\lambda) = \lim_{N\to\infty} \frac{1}{N!} \left[\lambda^{-1} (N\lambda^{-1} L)^{N+1} e^{-N\lambda^{-1} L} \right].$$

Assume that the heat semigroup $(e^{-tL})_{t>0}$ satisfies the classical $L^p - L^2$ estimates

$$\|e^{-tL}\|_{L^p \to L^2} \le Ct^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \qquad \text{for every } t > 0 \text{ and some } p \in [1,2].$$

Then we observe that for every integer $N \geq 3$

$$\begin{aligned} \|L^N e^{-tL}\|_{L^p \to L^{p'}} &\le \|e^{-\frac{t}{N}L}\|_{L^2 \to L^{p'}} \|L^N e^{-t(1-\frac{2}{N})L}\|_{L^2 \to L^2} \|e^{-\frac{t}{N}L}\|_{L^p \to L^2} \\ &\le C\left(\frac{t}{N}\right)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} \left(\frac{N}{t(1-\frac{2}{N})}\right)^N e^{-N} \\ &\le Ct^{-N-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} N^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} (Ne^{-1})^N \\ &\le Ct^{-N-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} N^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} (N-1)!\sqrt{N}, \end{aligned}$$

(use Stirling's formula for the last inequality).

Assume that the heat semigroup $(e^{-tL})_{t>0}$ satisfies the classical $L^p - L^2$ estimates

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Then we observe that for every integer $N \ge 3$

$$\begin{aligned}
\|L^N e^{-tL}\|_{L^p\to L^{p'}} &\le \|e^{-\frac{t}{N}L}\|_{L^2\to L^{p'}} \|L^N e^{-t(1-\frac{2}{N})L}\|_{L^2\to L^2} \|e^{-\frac{t}{N}L}\|_{L^p\to L^2} \\
&\le C\left(\frac{t}{N}\right)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} \left(\frac{N}{t(1-\frac{2}{N})}\right)^N e^{-N} \\
&\le Ct^{-N-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} N^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} (Ne^{-1})^N \\
&\le Ct^{-N-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} N^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} (N-1)!\sqrt{N},
\end{aligned}$$

(use Stirling's formula for the last inequality).
Therefore we see that the gap between this very general estimate and the one required in the previous theorem is an extra term of order $N^{\frac{1}{2}}$.

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Then we observe that for every integer $N \ge 3$

$$\begin{aligned}
\|L^N e^{-tL}\|_{L^p\to L^{p'}} &\le \|e^{-\frac{t}{N}L}\|_{L^2\to L^{p'}}\|L^N e^{-t(1-\frac{2}{N})L}\|_{L^2\to L^2}\|e^{-\frac{t}{N}L}\|_{L^p\to L^2}\\
&\le C\left(\frac{t}{N}\right)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)}\left(\frac{N}{t(1-\frac{2}{N})}\right)^{N} e^{-N}\\
&\le Ct^{-N-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)}N^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)}(Ne^{-1})^N\\
&\le Ct^{-N-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)}N^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)}(N-1)!\sqrt{N},
\end{aligned}$$

(use Stirling's formula for the last inequality).
Therefore we see that the gap between this very general estimate and the one required in the previous theorem is an extra term of order $N^{\frac{1}{2}}$.
Example: if L satisfies the dispersive estimate

$$\|e^{itL}\|_{1\to\infty} \le C|t|^{-d/2}, \quad t\in\mathbb{R},\ t\neq 0$$

then it satisfies assertion 2) of the previous theorem, namely

$$\|L^N e^{-tL}\|_{L^p\to L^{p'}} \le C(N-1)!N^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}t^{-N-\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}.$$