Functional calculus of Dirac operators and tent spaces

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joint work with D. Frey and A. McIntosh (ANU)

Besançon, October 2014

BISECTORIAL OPERATORS

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2. $X = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n)$.
$A = \begin{pmatrix} 0 & -divB \\ \nabla & 0 \end{pmatrix}$.
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Q: Conditions on A to have a bounded H^∞ functional calculus?

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\|(t,x) \mapsto Q_t^B u(x)\|_{L^2(L^2)} &\lesssim \|(t,x) \mapsto Q_t^B(I - P_t)u(x)\|_{L^2(L^2)} \\
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\end{aligned}$$

Functional calculus in L^2

[Axelsson, Keith, McIntosh Inv. Math. 06] Π_B has a bounded H^∞ functional calculus in L^2 if and only if

$$\left(\int_0^\infty \|Q_t^B u\|_2^2 \frac{dt}{t}\right)^{\frac{1}{2}} \sim \|u\|_2 \quad \forall u \in L^2,$$

where $Q_t^B = t\Pi_B(I + t^2\Pi_B^2)^{-1}$, $P_t = (I + t^2\Pi^2)^{-1}$,
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$$\begin{aligned}\|(t,x) \mapsto Q_t^B u(x)\|_{L^2(L^2)} &\lesssim \|(t,x) \mapsto Q_t^B(I - P_t)u(x)\|_{L^2(L^2)}\\ &+\|(t,x) \mapsto (Q_t^B - \gamma_t A_t)P_t u(x)\|_{L^2(L^2)} \text{ (}T(1)\text{ reduction)}\\ &+\|(t,x) \mapsto \gamma_t A_t(I - P_t)u(x)\|_{L^2(L^2)} \text{ (dyadic to continuous)}\\ &+\|(t,x) \mapsto \gamma_t A_t u(x)\|_{L^2(L^2)} \text{ (Carleson's measure estimate)}\end{aligned}$$

$$\|(t,x) \mapsto \gamma_t A_t u(x)\|_{L^2(L^2)} \lesssim \|u\|_2 \||\gamma_t(x)|^2 \frac{dtdx}{t}\|_{Car}$$

FUNC. CALC. IN L^p: VERTICAL SQUARE FUNCTIONS

[Hytönen, McIntosh, P., JFA 08] Π_B has a bounded H^∞ functional calculus if and only if

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FUNC. CALC. IN L^p: CONICAL SQUARE FUNCTIONS

[Frey, McIntosh, P., 14]

Π_B has a bounded H^∞ functional calculus in L^p if

$$\|(t,x) \mapsto (Q_t^B)^{4n} u(x)\|_{T^{p,2}} \sim \|u\|_p \quad \forall u \in L^p,$$

where [Coifman, Meyer, Stein, JFA 85]

$$\|F\|_{T^{p,2}} := (\int_{\mathbb{R}^n} (\int_0^\infty t^{-n} \int_{B(x,t)} |F(y,t)|^2 \frac{dtdy}{t})^{\frac{p}{2}} dx)^{\frac{1}{p}}.$$

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TENT BOUNDEDNESS

$\{T_t,\ t \geq 0\} \subset B(L^2)$.

$$\|(x,t) \mapsto T_t F(.,t)(x)\|_{L^p(L^2)} \lesssim \|F\|_{L^p(L^2)} \quad \forall F \in L^p(L^2)$$

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i.e. for all Borel sets $E, F \subset \mathbb{R}^n$

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Key remark: $A = -i\frac{d}{dx}$ on $L^p(\mathbb{R})$. $T_t = exp(itA)$.
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RESTRICTIONS IN p

Under the L^2 assumptions, for all $p \in (1,\infty)$,

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Calderón reproducing formula:

$$(Q_t^B)^{4n}(I-(P_t)^{4n})u = C\int_0^\infty ((Q_t^B)^{4n}(I-(P_t)^{4n})Q_s^{4n})Q_s^{4n}u\frac{ds}{s}.$$

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Integral operators on tent spaces:

$$\int_0^\infty ((Q_t^B)^{4n}(I-(P_t)^{4n})Q_s^{4n})Q_s^{4n}u =: T_K((s,x) \mapsto Q_s^{4n}u(x))\frac{ds}{s}.$$

Singular integral operator theory on tent spaces

$$T_K F(x,t) = \int_0^\infty K(t,s)F(.,s)(x)ds,$$

where $K(t,s) \in B(L^2)$.
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Example [FMcP14]: If $T_K \in B(T^{p,2})$ and $\{K(t,s)\ ;\ t,s > 0\}$ has $L^p - L^2$ off-diagonal bounds of any order, then $T_{K_1} F(x,t) = \int_0^\infty \min(\frac{t}{s}, \frac{s}{t}) K(t,s) F(.,s)(x) ds$, is bounded of $T^{q,2}$ for $q \in (p_*, p]$.

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