Hypercontractivity and free products

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Non commutative analysis

$\mathcal{M} \subset B(\ell_2)$ von Neumann algebra with a normal faithful trace τ

$$\begin{array}{ccc}
(\mathcal{M},\tau) & \leftrightarrow & (L_\infty(\Omega),\mu) \\
\tau(xy)=\tau(yx) & \leftrightarrow & fg=gf \\
\text{normal} & \leftrightarrow & \text{dominated cv thm} \\
\text{faithful} & \leftrightarrow & \text{full support} \\
L_0(\mathcal{M},\tau) & \leftrightarrow & L_0(\Omega,\mu)
\end{array}$$

$f:\mathbb{R}\to\mathbb{R}$, $x=x^*\in L_0(\mathcal{M},\tau)$, $y\in L_0(\mathcal{M},\tau)$

$$\text{Functional calculus : } f(x)\in L_0(\mathcal{M},\tau)$$

$$|y|=(y^*y)^{\frac{1}{2}}$$

$$L_p(\mathcal{M},\tau)=\{x\in L_0(\mathcal{M},\tau)\mid \|x\|_p^p=\tau(|x|^p)<\infty\}$$

Examples

$(\mathcal{M},\tau) = (\mathbb{M}_n, \mathrm{Tr}\,)$: $L_p(\mathbb{M}_n, \mathrm{Tr}\,) = S_p^n$ p-Schatten Class

$(\mathcal{M},\tau) = (\mathbb{B}(\ell_2), \mathrm{Tr}\,)$: $L_p(\mathbb{B}(\ell_2), \mathrm{Tr}\,) = S_p$ p-Schatten Class

G discete group : left representation $\lambda : G \to \mathbb{B}(\ell_2 G)$

$$L(G) = \lambda(G)'' = \overline{\mathrm{Span}\,\{\lambda(g)\,;g \in G\}}^{w}$$

$$\tau(\lambda(g)) = \delta_{g=e} = \langle \delta_e, \lambda(g)\delta_e\rangle_{\ell_2 G}$$

$$L_p(L(G),\tau) = L_p(\hat{G})$$

Examples

G abelian, $\hat{G}$ compact :

$$\begin{array}{ccc} (L(G),\tau) & \leftrightarrow & L_\infty(\hat{G},\mu) \\ x=\sum c_g\lambda(g) & \leftrightarrow & f(\gamma)=\sum c_g\langle g,\gamma\rangle \\ L_p(L(G),\tau) & \leftrightarrow & L_p(\hat{G},\mu) \end{array}$$

$\mathbb{Z}$ (generated by z), $\mathbb{Z}/2\mathbb{Z}\,(\varepsilon)$, $(\mathbb{Z}/2\mathbb{Z})^n\,(\varepsilon_i)$, ...

$$L_p(\mathbb{Z}/2\mathbb{Z})=L_p(\{\pm 1\})$$

Free products :
$G_i=\mathbb{Z}/2\mathbb{Z}$, ε_i generators

$$G_1 * G_2=\{e,\ \varepsilon_1,\ \varepsilon_2,\ \varepsilon_1\varepsilon_2,\ \varepsilon_2\varepsilon_1,\ \varepsilon_1\varepsilon_2\varepsilon_1,\ ...\}$$

$\mathbb{Z}*\mathbb{Z}...*\mathbb{Z}=\mathbb{F}_n$ generated by $g_1,...,g_n$:
Length in $\mathbb{F}_n$ or $\mathbb{Z}/2\mathbb{Z}*...*\mathbb{Z}/2\mathbb{Z}$: $|g_1g_2^5g_1^{-2}|=8$.

Some difficulties

- Functions do not commute !
- Computations are more intricate (No Taylor expansion)
- Inegalities are much more difficult to obtain
- Some inequalities are false !

$$\nexists C > 0, \qquad \| |x| - |y| \|_1 \leqslant C \|x - y\|_1$$

- New phenomenons
- Very few tools are available

Freeness

Corresponds to independance in classical analysis

$$(\mathcal{M}_i, \tau_i) \quad \rightarrow \quad (\mathcal{M}_1, \tau_1), (\mathcal{M}_2, \tau_2) \subset (\mathcal{M}_1 * \mathcal{M}_2, \tau)$$

$\mathcal{M}_1 * \mathcal{M}_2$ free algebra over $\mathcal{M}_i$ with amalgamation $1_{\mathcal{M}_1} = 1_{\mathcal{M}_2}$:

$$x_1 y_1 \dots x_n y_n, \qquad x_i \in \mathcal{M}_1,\ y_i \in \mathcal{M}_2$$

$$\tau(x_1 y_1 \dots x_n y_n) = 0 \text{ if } \forall i,\ \tau_1(x_i) = \tau_2(y_i) = 0$$

Example : $$\tau(x_1 y_1 x_2) = \tau(x_1 x_2)\tau(y_1)$$

$$\tau(x_1 y_2 x_2 y_2) = \tau(x_1 x_2)\tau(y_1)\tau(y_2) + \tau(x_1)\tau(x_2)\tau(y_1 y_2) - \tau(x_1)\tau(x_2)\tau(y_1)\tau(y_2)$$

When $\mathcal{M}_i = L(G_i)$: $(L(G_1), \tau) * (L(G_2), \tau) = L(G_1 * G_2, \tau)$

$$(L(\mathbb{Z}/2\mathbb{Z}^{*n}), \tau) = (L_\infty(\{\pm 1\}), \mu)^{*n}$$

Freeness

Free Central Limit Theorem (Voiculescu)

$X_i = X_i^* \in \mathcal{M}_i$ are i.d. free random variables with $\tau(X_i) = 0$, $\tau(X_i^2) = 1$,

then $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ converges in moments to a semicircular variable c.

Law of c : $\tau(c^k) = \frac{1}{2\pi} \int_{-2}^{2} x^k \sqrt{4 - x^2} dx$

Freeness	$\leftrightarrow$	Independance
semicircle law	$\leftrightarrow$	normal law
Tchebychev's Polynomials u_n	$\leftrightarrow$	Hermite's Polynomials h_n
$\Gamma_0(R^n) = \{c_1, ..., c_n\}''$	$\leftrightarrow$	$\{g_1, ..., g_n\}'' = L_\infty(\mathbb{R}^n, \gamma_n)$
$(\Gamma_0(R^n), \tau) = L_\infty([-2, 2], dw)^{*n}$	$\leftrightarrow$	$L_\infty(\mathbb{R}^n, \gamma_n) = L_\infty(\mathbb{R}, \gamma)^{\otimes n}$

Markov Semigroups

$P_t : \mathcal{M} \to \mathcal{M}$ w^*-continuous semigroup with

- P_t unital
- P_t (completely) positive : $x \geqslant 0 \;\Rightarrow\; P_t(x) \geqslant 0$
- P_t trace preserving : $\tau(P_t(x)) = \tau(x)$

$$\|P_t : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{M}, \tau)\| = 1$$

Markov semigroup are preserved by tensor products and free products

Question of hypercontractivity :
Best $t_{p,q}$ so that $\|P_t : L_p(\mathcal{M}, \tau) \to L_q(\mathcal{M}, \tau)\| \leqslant 1 \qquad (p < q)$.

Question of ultracontractivity :
Best t_p so that $\|P_t : L_p(\mathcal{M}, \tau) \to L_\infty(\mathcal{M}, \tau)\| \leqslant 1$.

Examples of Markov Semigroups

Classical Poisson semigroups in harmonic analysis :

$$\begin{array}{ccc} \mathbb{Z} & : & P_t(z^n) = e^{-t|n|} z^n \\ \mathbb{Z}^d & : & P_t(z_1^{n_1} \dots z_d^{n_d}) = e^{-t(|n_1|+\dots+|n_d|)} z_1^{n_1} \dots z_d^{n_d} \\ \mathbb{Z}/2\mathbb{Z} & : & P_t(\varepsilon) = e^{-t}\varepsilon \\ (\mathbb{Z}/2\mathbb{Z})^n & : & P_t(\varepsilon_{i_1} \dots \varepsilon_{i_p}) = e^{-tp} \varepsilon_{i_1} \dots \varepsilon_{i_p} \end{array}$$

Non commutative Poisson semigroups

$$\begin{array}{ccc} \mathbb{F}_d & : & P_t(g_{i_1}^{n_1} \dots g_{i_l}^{n_l}) = e^{-t(|n_1|+\dots+|n_l|)} g_{i_1}^{n_1} \dots g_{i_l}^{n_l} \\ (\mathbb{Z}/2\mathbb{Z})^{*n} & : & P_t(\varepsilon_{i_1} \dots \varepsilon_{i_l}) = e^{-tl} \varepsilon_{i_1} \dots \varepsilon_{i_l} \end{array} \qquad i_1 \neq i_2 \dots \neq i_l$$

More generally $P_t(g) = e^{-t|g|} g$.
$P_t = e^{-Nt}$ where N is the number operator $N(g) = |g| g$.

Examples of Markov Semigroups

Ornstein-Ulhenbeck semigroup $P_t : L_\infty(\mathbb{R}^d, \gamma_d) \to L_\infty(\mathbb{R}^d, \gamma_d)$
$P_t = e^{-tA}$ where A satisfies

$$A(h_{k_1}(x_1)...h_{k_d}(x_d)) = (k_1 + ... + k_d)h_{k_1}(x_1)...h_{k_d}(x_d)$$

$$P_{t,\mathbb{R}^d} = P_{t,\mathbb{R}}^{\otimes d}$$

Free Ornstein-Ulhenbeck semigroup $P_t : \Gamma_0(\mathbb{R}^d) \to \Gamma_0(\mathbb{R}^d)$
$P_t = e^{-tA}$ where A satisfies

$$A(u_{k_1}(c_{i_1})...u_{k_l}(c_{i_l})) = (k_1 + ... + k_l)u_{k_1}(c_{i_1})...u_{k_l}(c_{i_l})$$

$$P_{t,\Gamma_0(\mathbb{R}^d)} = P_{t,\Gamma_0(\mathbb{R})}^{*d}$$

Examples of Markov Semigroups

Fermionic Ornstein-Ulhenbeck (or Poisson) semigroup

Fermions : $\omega_i^2 = 1$, $\omega_i\omega_j = -\omega_j\omega_i$

$\Gamma_{-1}(\mathbb{R}^n) = \{\omega_i, i = 1...d\}'' = \mathbb{M}_k$ for some k

Similarly to the Walsh system

$$\mathbb{M}_k \ni x = \sum_{A=\{i_1<...<i_l\}\subset[1,d]} \alpha_A \omega_{i_1}...\omega_{i_l}$$

$$A(\omega_{i_1}...\omega_{i_l}) = l\omega_{i_1}...\omega_{i_l}$$

$$P_t = e^{-At}$$

Khintchine's inequalities

Let W_n be the set of words of length n in $\mathbb{F}_d$

Haagerup

$$\Big\| \sum_{w \in W_n} \alpha_w \lambda(w) \Big\|_{L_\infty(\mathbb{F}_d)} \leqslant (n+1) \Big\| \sum_{w \in W_n} \alpha_w \lambda(w) \Big\|_{L_2(\mathbb{F}_d)}$$

Consequence for the Poisson semigroup :

Ultracontractivity

For any $t > 0$, $P_t : L_1(\mathbb{F}_d) \to L_\infty(\mathbb{F}_d)$

Similar results for the free Ornstein-Ulenbeck semigroup (Krolak),...

Commutative results

Bonami-Gross-Beckner : 2-points inequality

$$\forall\, a, b \in \mathbb{R}, \quad \left(\frac{|a+rb|^q + |a-rb|^q}{2}\right)^{\frac{1}{q}} \leqslant \left(\frac{|a+b|^p + |a-b|^p}{2}\right)^{\frac{1}{p}}$$

$$\Leftrightarrow \; r^2 \leqslant \frac{p-1}{q-1}$$

Hypercontractivity for $\mathbb{Z}/2\mathbb{Z}$, $p < q$

$$\|P_t : L_p(\{\pm 1\}) \to L_q(\{\pm 1\}\| \leqslant 1 \; \Leftrightarrow \; t \geqslant t_{p,q} = \frac{1}{2} \ln \frac{q-1}{p-1}$$

$$\left(\frac{|a+b|^p + |a-b|^p}{2}\right)^{\frac{1}{p}} = \|a + b\varepsilon\|_p$$

Corollary : Best constants in Khintchine's inequalities

Tensor product

If $\| T_i : L_p(\Omega_i) \to L_q(\Omega_i)\| = 1$ then

$$\| T_1 \otimes T_2 ; L_p(\Omega_1 \times \Omega_2) \to L_q(\Omega_1 \times \Omega_2)\| = 1$$

This is just Minkowski's inequality.

Hypercontractivity for $(\mathbb{Z}/2\mathbb{Z})^n$, $p < q$

$$\|P_t : L_p(\{\pm 1\}^n) \to L_q(\{\pm 1\}^n\| \leqslant 1 \ \Leftrightarrow\ t \geqslant t_{p,q} = \frac{1}{2} \ln \frac{q-1}{p-1}$$

Via TCL :

$$\frac{\varepsilon_1 + \ldots + \varepsilon_1}{\sqrt{n}} \to g$$

$$P_t\Big(\Big(\frac{\varepsilon_1 + \ldots + \varepsilon_1}{\sqrt{n}}\Big)^2\Big) = 1 + e^{-2t} \sum_{i \neq j} \frac{\varepsilon_i \varepsilon_j}{n}$$

$$P_t(g^2 - 1) = e^{-2t} P_t(g^2 - 1)$$

Hypercontractivity for the Ornstein-Ulhenbeck semigroup

$$\|P_t : L_p(\mathbb{R}^n, \gamma_n) \to L_q(\mathbb{R}^n, \gamma_n)\| \leqslant 1 \iff t \geqslant t_{p,q}$$

This is an example of diffusion semigroup
More elaborated tools $\to$ Emery, Bakry...

The Poisson semigroup on $\mathbb{Z}$ is not a diffusion semigroup

Hypercontractivity for $\mathbb{Z}$ or $\mathbb{Z}^n$, $p < q$ (Weissler)

$$\|P_t : L_p(\mathbb{T}) \to L_q(\mathbb{T})| \leqslant 1 \;\Leftrightarrow\; t \geqslant t_{p,q}$$

Different approach, pb : to distinguish z and $\overline{z}$

Gross, Olkiewicz and Zegarlinski (noncommutative case)

Hypercontractivity for all $p < q$ with time $t_{p,q} \Leftrightarrow (SL_2)$ inequality

$$(SL_p) \quad f > 0, \qquad \int f^q \ln f \, d\mu \leqslant \frac{q}{2(q-1)} \int f^{q-1} A(f) d\mu + \|f\|_q \ln \|f\|_q$$

Rq : L_p-L_2 hypercontractivity $p \in (2-\varepsilon, 2)$ with time $t_{p,2} \Rightarrow$ all $p < q$

Rq : $(SL_p) \Rightarrow (SL_q)$ if $p < q$

Non commutative approach

Ball-Carlen-Lieb : 2-points inequality

For $1 < p < 2$, $\forall\, a, b \in S_p$

$$\left(\frac{\|a+b\|_p^p + \|a-b\|_p^p}{2}\right)^{\frac{1}{p}} \geqslant \left(\|a\|_p^2 + (p-1)\|b\|_p^2\right)^{\frac{1}{2}}$$

Proof works only for matrix algebras

with Xu :

For $1 < p < 2$, $\forall\, x \in \mathcal{M}$ and $\mathbb{E} : \mathcal{M} \to \mathcal{M}$ a conditional expectation

$$\|x\|_p^2 \geqslant \|\mathbb{E}x\|_p^2 + (p-1)\|x - \mathbb{E}x\|_p^2$$

Using an iteration argument

Hypercontractivity for fermions, $p < 2$ (Carlen-Lieb)

$$\|P_t : L_p(\Gamma_{-1}(\mathbb{R}^n)) \to L_2(\Gamma_{-1}(\mathbb{R}^n))\| \leqslant 1 \;\Leftrightarrow\; t \leqslant t_{2,p}$$

Using a smart Central Limit Theorem by Speicher

Hypercontractivity for free-semicirculars, $p < 2$ (Biane)

$$\|P_t : L_p(\Gamma_0(\mathbb{R}^n)) \to L_2(\Gamma_0(\mathbb{R}^n))\| \leqslant 1 \;\Leftrightarrow \geqslant t_{2,p}$$

Hypercontractivity for $L((\mathbb{Z}/2\mathbb{Z})^{*n})$, $p < 2$ (JPPPR)

$$\|P_t : L_p((\widehat{\mathbb{Z}/2\mathbb{Z})}^{*n}) \to L_2((\widehat{\mathbb{Z}/2\mathbb{Z})}^{*n})\| \leqslant 1 \;\Leftrightarrow\; t \geqslant t_{2,p}$$

$L(\mathbb{Z}^n)$?

First attempt (JPPPR)

$$\|P_t : L_p(\widehat{\mathbb{Z}^{*n}}) \to L_2(\widehat{\mathbb{Z}^{*n}})\| \leqslant 1 \;\Leftarrow\; t \geqslant 2t_{2,p}$$

Pf : $g_i \sim \varepsilon_{2i}\varepsilon_{2i+1}$ in distribution where ε_j are free
The pb is still to distinguish g_i and $g_i^{-1} \to$ factor 2.
Result is ok for "symmetric words in g_i and g_i^{-1}"

$z \sim \begin{bmatrix} 0 & \varepsilon_1 \\ \varepsilon_2 & 0 \end{bmatrix} \quad \to$ Weissler's result via 2-points the nc inequality

Second attempt (JPPPR)

$$\|P_t : L_p(\widehat{\mathbb{Z}^{*n}}) \to L_2(\widehat{\mathbb{Z}^{*n}})\| \leqslant 1 \;\Leftarrow\; t \geqslant 1.17\, t_{2,p}$$

Third attempt (with Xu)

For $p \leqslant \frac{4}{3}$
$$\|P_t : L_p(\widehat{\mathbb{Z}^{*n}}) \to L_2(\widehat{\mathbb{Z}^{*n}})\| \leqslant 1 \;\Leftrightarrow\; t \geqslant t_{2,p}$$

Ideas use the general 2-points inequality to project on symmetric word and make a correction

Optimal Khintchine's inequality

$$\Big\| \sum_{w \in W_n} \alpha_w \lambda(w) \Big\|_{L_4(\mathbb{F}_d)} \leqslant (n+1)^{\frac{1}{4}} \Big\| \sum_{w \in W_n} \alpha_w \lambda(w) \Big\|_{L_2(\mathbb{F}_d)}$$

Rq : impossible to reach 2 with this method