

Hypercontractivity and free products

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Non commutative analysis

$\mathcal{M} \subset B(\ell_2)$ von Neumann algebra with a normal faithful trace τ

$$\begin{array}{ll} (\mathcal{M}, \tau) & \leftrightarrow (L_\infty(\Omega), \mu) \\ \tau(xy) = \tau(yx) & \leftrightarrow fg = gf \\ \text{normal} & \leftrightarrow \text{dominated cv thm} \\ \text{faithful} & \leftrightarrow \text{full support} \\ L_0(\mathcal{M}, \tau) & \leftrightarrow L_0(\Omega, \mu) \end{array}$$

$f : \mathbb{R} \rightarrow \mathbb{R}, x = x^* \in L_0(\mathcal{M}, \tau), y \in L_0(\mathcal{M}, \tau)$

Functional calculus : $f(x) \in L_0(\mathcal{M}, \tau)$
 $|y| = (y^*y)^{\frac{1}{2}}$

$$L_p(\mathcal{M}, \tau) = \{x \in L_0(\mathcal{M}, \tau) \mid \|x\|_p^p = \tau(|x|^p) < \infty\}$$

Examples

$$\begin{aligned}(\mathcal{M}, \tau) = (\mathbb{M}_n, \text{Tr}) : & \quad L_p(\mathbb{M}_n, \text{Tr}) = S_p^n \text{ } p\text{-Schatten Class} \\(\mathcal{M}, \tau) = (\mathbb{B}(\ell_2), \text{Tr}) : & \quad L_p(\mathbb{B}(\ell_2), \text{Tr}) = S_p \text{ } p\text{-Schatten Class}\end{aligned}$$

G discrete group : left representation $\lambda : G \rightarrow \mathbb{B}(\ell_2 G)$

$$L(G) = \lambda(G)'' = \overline{\text{Span} \{ \lambda(g) ; g \in G \}}^w$$

$$\tau(\lambda(g)) = \delta_{g=e} = \langle \delta_e, \lambda(g) \delta_e \rangle_{\ell_2 G}$$

$$L_p(L(G), \tau) = L_p(\hat{G})$$

Examples

G abelian, \hat{G} compact :

$$\begin{aligned} (L(G), \tau) &\leftrightarrow L_\infty(\hat{G}, \mu) \\ x = \sum c_g \lambda(g) &\leftrightarrow f(\gamma) = \sum c_g \langle g, \gamma \rangle \\ L_p(L(G), \tau) &\leftrightarrow L_p(\hat{G}, \mu) \end{aligned}$$

\mathbb{Z} (generated by z), $\mathbb{Z}/2\mathbb{Z}(\varepsilon)$, $(\mathbb{Z}/2\mathbb{Z})^n(\varepsilon_i), \dots$

$$L_p(\mathbb{Z}/2\mathbb{Z}) = L_p(\{\pm 1\})$$

Free products :

$G_i = \mathbb{Z}/2\mathbb{Z}$, ε_i generators

$$G_1 * G_2 = \{e, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2, \varepsilon_2\varepsilon_1, \varepsilon_1\varepsilon_2\varepsilon_1, \dots\}$$

$\mathbb{Z} * \mathbb{Z} \dots * \mathbb{Z} = \mathbb{F}_n$ generated by g_1, \dots, g_n :

Length in \mathbb{F}_n or $\mathbb{Z}/2\mathbb{Z} * \dots * \mathbb{Z}/2\mathbb{Z}$: $|g_1 g_2^5 g_1^{-2}| = 8$.

Some difficulties

- Functions do not commute!
- Computations are more intricate (No Taylor expansion)
- Inequalities are much more difficult to obtain
- Some inequalities are false!

$$\nexists C > 0, \quad |||x| - |y|||_1 \leq C \|x - y\|_1$$

- New phenomena
- Very few tools are available

Freeness

Corresponds to independence in classical analysis

$$(\mathcal{M}_i, \tau_i) \rightarrow (\mathcal{M}_1, \tau_1), (\mathcal{M}_2, \tau_2) \subset (\mathcal{M}_1 * \mathcal{M}_2, \tau)$$

$\mathcal{M}_1 * \mathcal{M}_2$ free algebra over \mathcal{M}_i with amalgamation $1_{\mathcal{M}_1} = 1_{\mathcal{M}_2}$:

$$x_1 y_1 \dots x_n y_n, \quad x_i \in \mathcal{M}_1, y_i \in \mathcal{M}_2$$

$$\tau(x_1 y_1 \dots x_n y_n) = 0 \text{ if } \forall i, \tau_1(x_i) = \tau_2(y_i) = 0$$

Example :

$$\tau(x_1 y_1 x_2) = \tau(x_1 x_2) \tau(y_1)$$

$$\tau(x_1 y_2 x_2 y_2) = \tau(x_1 x_2) \tau(y_1) \tau(y_2) + \tau(x_1) \tau(x_2) \tau(y_1 y_2) - \tau(x_1) \tau(x_2) \tau(y_1) \tau(y_2)$$

When $\mathcal{M}_i = L(G_i) : (L(G_1), \tau) * (L(G_2), \tau) = L(G_1 * G_2, \tau)$

$$(L(\mathbb{Z}/2\mathbb{Z}^{*n}), \tau) = (L_\infty(\{\pm 1\}), \mu)^{*n}$$

Free Central Limit Theorem (Voiculescu)

$X_i = X_i^* \in \mathcal{M}_i$ are i.d. free random variables with $\tau(X_i) = 0$, $\tau(X_i^2) = 1$, then $\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ converges in moments to a semicircular variable c .

$$\text{Law of } c : \tau(c^k) = \frac{1}{2\pi} \int_{-2}^2 x^k \sqrt{4-x^2} dx$$

Freeness	\leftrightarrow	Independance
semicircle law	\leftrightarrow	normal law
Tchebychev's Polynomials u_n	\leftrightarrow	Hermite's Polynomials h_n
$\Gamma_0(\mathbb{R}^n) = \{c_1, \dots, c_n\}''$	\leftrightarrow	$\{g_1, \dots, g_n\}'' = L_\infty(\mathbb{R}^n, \gamma_n)$
$(\Gamma_0(\mathbb{R}^n), \tau) = L_\infty([-2, 2], dw)^{*n}$	\leftrightarrow	$L_\infty(\mathbb{R}^n, \gamma_n) = L_\infty(\mathbb{R}, \gamma)^{\otimes n}$

Markov Semigroups

$P_t : \mathcal{M} \rightarrow \mathcal{M}$ w^* -continuous semigroup with

- P_t unital
- P_t (completely) positive : $x \geq 0 \Rightarrow P_t(x) \geq 0$
- P_t trace preserving : $\tau(P_t(x)) = \tau(x)$

$$\|P_t : L_p(\mathcal{M}, \tau) \rightarrow L_p(\mathcal{M}, \tau)\| = 1$$

Markov semigroup are preserved by tensor products and free products

Question of hypercontractivity :

Best $t_{p,q}$ so that $\|P_t : L_p(\mathcal{M}, \tau) \rightarrow L_q(\mathcal{M}, \tau)\| \leq 1$ ($p < q$).

Question of ultracontractivity :

Best t_p so that $\|P_t : L_p(\mathcal{M}, \tau) \rightarrow L_\infty(\mathcal{M}, \tau)\| \leq 1$.

Examples of Markov Semigroups

Classical Poisson semigroups in harmonic analysis :

$$\begin{aligned}
 \mathbb{Z} & : P_t(z^n) = e^{-t|n|} z^n \\
 \mathbb{Z}^d & : P_t(z_1^{n_1} \dots z_d^{n_d}) = e^{-t(|n_1| + \dots + |n_d|)} z_1^{n_1} \dots z_d^{n_d} \\
 \mathbb{Z}/2\mathbb{Z} & : P_t(\varepsilon) = e^{-t}\varepsilon \\
 (\mathbb{Z}/2\mathbb{Z})^n & : P_t(\varepsilon_{i_1} \dots \varepsilon_{i_p}) = e^{-tp} \varepsilon_{i_1} \dots \varepsilon_{i_p}
 \end{aligned}$$

Non commutative Poisson semigroups

$$\begin{aligned}
 \mathbb{F}_d & : P_t(g_{i_1}^{n_1} \dots g_{i_l}^{n_l}) = e^{-t(|n_1| + \dots + |n_l|)} g_{i_1}^{n_1} \dots g_{i_l}^{n_l} \quad i_1 \neq i_2 \dots \neq i_l \\
 (\mathbb{Z}/2\mathbb{Z})^{*n} & : P_t(\varepsilon_{i_1} \dots \varepsilon_{i_l}) = e^{-tl} \varepsilon_{i_1} \dots \varepsilon_{i_l}
 \end{aligned}$$

More generally $P_t(g) = e^{-t|g|} g$.

$P_t = e^{-Nt}$ where N is the number operator $N(g) = |g|g$.

Examples of Markov Semigroups

Ornstein-Uhlenbeck semigroup $P_t : L_\infty(\mathbb{R}^d, \gamma_d) \rightarrow L_\infty(\mathbb{R}^d, \gamma_d)$
 $P_t = e^{-tA}$ where A satisfies

$$A(h_{k_1}(x_1) \dots h_{k_d}(x_d)) = (k_1 + \dots + k_d) h_{k_1}(x_1) \dots h_{k_d}(x_d)$$

$$P_{t, \mathbb{R}^d} = P_{t, \mathbb{R}}^{\otimes d}$$

Free Ornstein-Uhlenbeck semigroup $P_t : \Gamma_0(\mathbb{R}^d) \rightarrow \Gamma_0(\mathbb{R}^d)$
 $P_t = e^{-tA}$ where A satisfies

$$A(u_{k_1}(c_{i_1}) \dots u_{k_l}(c_{i_l})) = (k_1 + \dots + k_l) u_{k_1}(c_{i_1}) \dots u_{k_l}(c_{i_l})$$

$$P_{t, \Gamma_0(\mathbb{R}^d)} = P_{t, \Gamma_0(\mathbb{R})}^{*d}$$

Examples of Markov Semigroups

Fermionic Ornstein-Uhlenbeck (or Poisson) semigroup

Fermions : $\omega_i^2 = 1, \omega_i\omega_j = -\omega_j\omega_i$

$\Gamma_{-1}(\mathbb{R}^n) = \{\omega_i, i = 1 \dots d\}'' = \mathbb{M}_k$ for some k

Similarly to the Walsh system

$$\mathbb{M}_k \ni x = \sum_{A=\{i_1 < \dots < i_l\} \subset [1, d]} \alpha_A \omega_{i_1} \dots \omega_{i_l}$$

$$A(\omega_{i_1} \dots \omega_{i_l}) = l \omega_{i_1} \dots \omega_{i_l}$$

$$P_t = e^{-At}$$

Khintchine's inequalities

Let W_n be the set of words of length n in \mathbb{F}_d

Haagerup

$$\left\| \sum_{w \in W_n} \alpha_w \lambda(w) \right\|_{L_\infty(\mathbb{F}_d)} \leq (n+1) \left\| \sum_{w \in W_n} \alpha_w \lambda(w) \right\|_{L_2(\mathbb{F}_d)}$$

Consequence for the Poisson semigroup :

Ultracontractivity

For any $t > 0$, $P_t : L_1(\mathbb{F}_d) \rightarrow L_\infty(\mathbb{F}_d)$

Similar results for the free Ornstein-Uhlenbeck semigroup (Krolak),...

Commutative results

Bonami-Gross-Beckner : 2-points inequality

$$\forall a, b \in \mathbb{R}, \quad \left(\frac{|a + rb|^q + |a - rb|^q}{2} \right)^{\frac{1}{q}} \leq \left(\frac{|a + b|^p + |a - b|^p}{2} \right)^{\frac{1}{p}}$$
$$\Leftrightarrow r^2 \leq \frac{p-1}{q-1}$$

Hypercontractivity for $\mathbb{Z}/2\mathbb{Z}$, $p < q$

$$\|P_t : L_p(\{\pm 1\}) \rightarrow L_q(\{\pm 1\})\| \leq 1 \Leftrightarrow t \geq t_{p,q} = \frac{1}{2} \ln \frac{q-1}{p-1}$$

$$\left(\frac{|a + b|^p + |a - b|^p}{2} \right)^{\frac{1}{p}} = \|a + b\varepsilon\|_p$$

Corollary : Best constants in Khintchine's inequalities

Tensor product

If $\|T_i : L_p(\Omega_i) \rightarrow L_q(\Omega_i)\| = 1$ then

$$\|T_1 \otimes T_2 : L_p(\Omega_1 \times \Omega_2) \rightarrow L_q(\Omega_1 \times \Omega_2)\| = 1$$

This is just Minkowski's inequality.

Hypercontractivity for $(\mathbb{Z}/2\mathbb{Z})^n$, $p < q$

$$\|P_t : L_p(\{\pm 1\}^n) \rightarrow L_q(\{\pm 1\}^n)\| \leq 1 \Leftrightarrow t \geq t_{p,q} = \frac{1}{2} \ln \frac{q-1}{p-1}$$

Via TCL :

$$\frac{\varepsilon_1 + \dots + \varepsilon_1}{\sqrt{n}} \rightarrow g$$

$$P_t\left(\left(\frac{\varepsilon_1 + \dots + \varepsilon_1}{\sqrt{n}}\right)^2\right) = 1 + e^{-2t} \sum_{i \neq j} \frac{\varepsilon_i \varepsilon_j}{n}$$

$$P_t(g^2 - 1) = e^{-2t} P_t(g^2 - 1)$$

Hypercontractivity for the Ornstein-Uhlenbeck semigroup

$$\|P_t : L_p(\mathbb{R}^n, \gamma_n) \rightarrow L_q(\mathbb{R}^n, \gamma_n)\| \leq 1 \Leftrightarrow t \geq t_{p,q}$$

This is an example of diffusion semigroup
More elaborated tools \rightarrow Emery, Bakry...

The Poisson semigroup on \mathbb{Z} is not a diffusion semigroup

Hypercontractivity for \mathbb{Z} or \mathbb{Z}^n , $p < q$ (Weissler)

$$\|P_t : L_p(\mathbb{T}) \rightarrow L_q(\mathbb{T})\| \leq 1 \Leftrightarrow t \geq t_{p,q}$$

Different approach, pb : to distinguish z and \bar{z}

Gross, Olkiewicz and Zegarlinski (noncommutative case)

Hypercontractivity for all $p < q$ with time $t_{p,q} \Leftrightarrow (SL_2)$ inequality

$$(SL_p) \quad f > 0, \quad \int f^q \ln f \, d\mu \leq \frac{q}{2(q-1)} \int f^{q-1} A(f) \, d\mu + \|f\|_q \ln \|f\|_q$$

Rq : L_p - L_2 hypercontractivity $p \in (2 - \varepsilon, 2)$ with time $t_{p,2} \Rightarrow$ all $p < q$

Rq : $(SL_p) \Rightarrow (SL_q)$ if $p < q$

Non commutative approach

Ball-Carlen-Lieb : 2-points inequality

For $1 < p < 2$, $\forall a, b \in S_p$

$$\left(\frac{\|a + b\|_p^p + \|a - b\|_p^p}{2} \right)^{\frac{1}{p}} \geq \left(\|a\|_p^2 + (p - 1)\|b\|_p^2 \right)^{\frac{1}{2}}$$

Proof works only for matrix algebras

with Xu :

For $1 < p < 2$, $\forall x \in \mathcal{M}$ and $\mathbb{E} : \mathcal{M} \rightarrow \mathcal{M}$ a conditional expectation

$$\|x\|_p^2 \geq \|\mathbb{E}x\|_p^2 + (p - 1)\|x - \mathbb{E}x\|_p^2$$

Using an iteration argument

Hypercontractivity for fermions, $p < 2$ (Carlen-Lieb)

$$\|P_t : L_p(\Gamma_{-1}(\mathbb{R}^n)) \rightarrow L_2(\Gamma_{-1}(\mathbb{R}^n))\| \leq 1 \Leftrightarrow t \leq t_{2,p}$$

Using a smart Central Limit Theorem by Speicher

Hypercontractivity for free-semicirculars, $p < 2$ (Biane)

$$\|P_t : L_p(\Gamma_0(\mathbb{R}^n)) \rightarrow L_2(\Gamma_0(\mathbb{R}^n))\| \leq 1 \Leftrightarrow t \geq t_{2,p}$$

Hypercontractivity for $L((\mathbb{Z}/2\mathbb{Z})^{*n})$, $p < 2$ (JPPPR)

$$\|P_t : L_p(\widehat{(\mathbb{Z}/2\mathbb{Z})^{*n}}) \rightarrow L_2(\widehat{(\mathbb{Z}/2\mathbb{Z})^{*n}})\| \leq 1 \Leftrightarrow t \geq t_{2,p}$$

$L(\mathbb{Z}^n)$?

First attempt (JPPPR)

$$\|P_t : L_p(\widehat{\mathbb{Z}^{*n}}) \rightarrow L_2(\widehat{\mathbb{Z}^{*n}})\| \leq 1 \iff t \geq 2t_{2,p}$$

Pf : $g_i \sim \varepsilon_{2i}\varepsilon_{2i+1}$ in distribution where ε_j are free

The pb is still to distinguish g_i and $g_i^{-1} \rightarrow$ factor 2.

Result is ok for “symmetric words in g_i and g_i^{-1} ”

$$z \sim \begin{bmatrix} 0 & \varepsilon_1 \\ \varepsilon_2 & 0 \end{bmatrix} \rightarrow \text{Weissler's result via 2-points the nc inequality}$$

Second attempt (JPPPR)

$$\|P_t : L_p(\widehat{\mathbb{Z}^{*n}}) \rightarrow L_2(\widehat{\mathbb{Z}^{*n}})\| \leq 1 \iff t \geq 1.17 t_{2,p}$$

Third attempt (with Xu)

For $p \leq \frac{4}{3}$

$$\|P_t : L_p(\widehat{\mathbb{Z}^{*n}}) \rightarrow L_2(\widehat{\mathbb{Z}^{*n}})\| \leq 1 \Leftrightarrow t \geq t_{2,p}$$

Ideas use the general 2-points inequality to project on symmetric word and make a correction

Optimal Khintchine's inequality

$$\left\| \sum_{w \in W_n} \alpha_w \lambda(w) \right\|_{L_4(\mathbb{F}_d)} \leq (n+1)^{\frac{1}{4}} \left\| \sum_{w \in W_n} \alpha_w \lambda(w) \right\|_{L_2(\mathbb{F}_d)}$$

Rq : impossible to reach 2 with this method