Conical square functions associated with Bessel, Laguerre and Schrödinger operators in UMD Banach spaces

Lourdes Rodríguez-Mesa

Departamento de Análisis Matemático,
Universidad de La Laguna
Spain

Workshop on Functional Calculus and Harmonic Analysis of Semigroups

Besançon, October 1-3, 2014

In collaboration with:

JORGE J. BETANCOR, JUAN C. FARIÑA and ALEJANDRO J. CASTRO
Departamento de Análisis Matemático
Universidad de La Laguna
Canary Islands, SPAIN

ARXIV: 1404.5779V1 (Preprint, 2014)

Let us consider $\mathbb{B}$ a Banach space, $(\Omega, \mathscr{M}, \mu)$ a measure space and $1 \leq p < \infty$.

$$L^p(\Omega, \mu; \mathbb{B}) = \left\{ f : \Omega \longrightarrow \mathbb{B} : \|f\|_{L^p(\Omega, \mathbb{B})} = \left(\int_\Omega \|f(x)\|_{\mathbb{B}}^p dx \right)^{1/p} < \infty \right\}$$

Question

To obtain equivalent norms for functions $f \in L^p(\Omega, \mu; \mathbb{B})$ by using conical square functions

1. Square functions associated to the Laplacian

2. Conical square functions associated to other operators
   - Schrödinger operator. Harmonic oscillator
   - Bessel and Laguerre operator

$$\Delta = \sum_{j=1}^{n} \partial_{x_j}^2 \quad \text{in } \mathbb{R}^n$$

- Littlewood-Paley square function

$$g_2(f)(x) = \left\{ \int_{\mathbb{R}^n} |t\partial_t P_t(f)(x)|^2 \frac{dt}{t} \right\}^{1/2}, \quad x \in \mathbb{R}^n.$$

- Conical square function

$$S_2(f)(x) = \left\{ \int_{\Gamma(x)} |t\partial_t P(f)(y)|^2 \frac{dydt}{t^{n+1}} \right\}^{1/2}, \quad x \in \mathbb{R}^n.$$

$$P_t(f)(x) = \int_{\mathbb{R}^n} c_n \frac{t}{(t^2 + |x-y|^2)^{(n+1)/2}} f(y)dy, \quad x \in \mathbb{R}^n, t > 0.$$

$$\Gamma(x) = \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |y-x| < t\}$$

Classical theorem (Stein, 1970)

Let $1 < p < \infty$ and denote by Ψ_2 any of the operators g_2 or S_2. Then, there exists $C > 0$ such that

$$\frac{1}{C}\|f\|_{L^p(\mathbb{R}^n)} \leq \|\Psi_2(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

From now on we write $R(f) \sim T(f)$ to indicate that there exists $C > 0$ such that

$$\frac{1}{C}R(f) \leq T(f) \leq CR(f), \quad \text{for all } f$$

Classical theorem (Stein, 1970)

Let $1 < p < \infty$ and denote by Ψ_2 any of the operators g_2 or S_2. Then, there exists $C > 0$ such that

$$\frac{1}{C}\|f\|_{L^p(\mathbb{R}^n)} \leq \|\Psi_2(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

From now on we write $R(f) \sim T(f)$ to indicate that there exists $C > 0$ such that

$$\frac{1}{C}R(f) \leq T(f) \leq CR(f), \quad \text{for all } f$$

- $g_{2,\mathbb{B}}(f)(x) = \left\{ \int_{\mathbb{R}^n} \left\| t\partial_t P_t(f)(x) \right\|_{\mathbb{B}}^2 \frac{dt}{t} \right\}^{1/2}, \quad x \in \mathbb{R}^n.$
- $S_{2,\mathbb{B}}(f)(x) = \left\{ \int_{\Gamma(x)} \left\| t\partial_t P(f)(y) \right\|_{\mathbb{B}}^2 \frac{dydt}{t^{n+1}} \right\}^{1/2}, \quad x \in \mathbb{R}^n.$

A Banach-valued version of the classical theorem

Let $1 < p < \infty$ and consider $\Psi_{2,\mathbb{B}}$ any of the operators $g_{2,\mathbb{B}}$ or $S_{2,\mathbb{B}}$. The following assertions are equivalent.

(i) $\mathbb{B}$ is isomorphic to a Hilbert space.

(ii) $\|\Psi_{2,\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$.

- *Xu*, J. Reine Angew. Math., 1998: $\Psi_{q,\mathbb{B}}$, martingale cotype $q \geq 2$ and martingale type $q \leq 2$ for $\mathbb{B}$
- *Martínez, Torrea and Xu*, Adv. Math., 2006: symmetric diffusion semigroups
- *Torrea and Zhang*, Proc. Roy. Soc. Edin., 2014: $t^\beta \partial_t^\beta P_t(f)$, $\beta > 0$

- $g_{2,\mathbb{B}}(f)(x) = \left\{ \int_{\mathbb{R}^n} \left\| t\partial_t P_t(f)(x) \right\|_{\mathbb{B}}^2 \frac{dt}{t} \right\}^{1/2}, \quad x \in \mathbb{R}^n.$
- $S_{2,\mathbb{B}}(f)(x) = \left\{ \int_{\Gamma(x)} \left\| t\partial_t P(f)(y) \right\|_{\mathbb{B}}^2 \frac{dydt}{t^{n+1}} \right\}^{1/2}, \quad x \in \mathbb{R}^n.$

A Banach-valued version of the classical theorem

Let $1 < p < \infty$ and consider $\Psi_{2,\mathbb{B}}$ any of the operators $g_{2,\mathbb{B}}$ or $S_{2,\mathbb{B}}$. The following assertions are equivalent.

(i) $\mathbb{B}$ is isomorphic to a Hilbert space.

(ii) $\|\Psi_{2,\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$.

- *Xu*, J. Reine Angew. Math., 1998: $\Psi_{q,\mathbb{B}}$, martingale cotype $q \geq 2$ and martingale type $q \leq 2$ for $\mathbb{B}$
- *Martínez, Torrea and Xu*, Adv. Math., 2006: symmetric diffusion semigroups
- *Torrea and Zhang*, Proc. Roy. Soc. Edin., 2014: $t^\beta \partial_t^\beta P_t(f)$, $\beta > 0$

- $g_{2,\mathbb{B}}(f)(x) = \left\{ \int_{\mathbb{R}^n} \left\| t\partial_t P_t(f)(x) \right\|_{\mathbb{B}}^2 \frac{dt}{t} \right\}^{1/2}, \quad x \in \mathbb{R}^n.$
- $S_{2,\mathbb{B}}(f)(x) = \left\{ \int_{\Gamma(x)} \left\| t\partial_t P(f)(y) \right\|_{\mathbb{B}}^2 \frac{dydt}{t^{n+1}} \right\}^{1/2}, \quad x \in \mathbb{R}^n.$

A Banach-valued version of the classical theorem

Let $1 < p < \infty$ and consider $\Psi_{2,\mathbb{B}}$ any of the operators $g_{2,\mathbb{B}}$ or $S_{2,\mathbb{B}}$. The following assertions are equivalent.

(i) $\mathbb{B}$ is isomorphic to a Hilbert space.

(ii) $\|\Psi_{2,\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$.

- *Xu*, J. Reine Angew. Math., 1998: $\Psi_{q,\mathbb{B}}$, martingale cotype $q \geq 2$ and martingale type $q \leq 2$ for $\mathbb{B}$
- *Martínez, Torrea and Xu*, Adv. Math., 2006: symmetric diffusion semigroups
- *Torrea and Zhang*, Proc. Roy. Soc. Edin., 2014: $t^\beta \partial_t^\beta P_t(f)$, $\beta > 0$

Fractional derivative (Segovia and Wheeden, J. Math. Mech., 1970)

Let $\beta > 0$ and m the smallest integer that strictly exceeds β.
Consider a nice enough complex function F defined on $\Omega \times (0, \infty)$.
The β-th derivative $\partial_t^\beta F$ is defined by

$$\partial_t^\beta F(x,t) = \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^m F(x, t+s) s^{m-\beta-1} ds, \quad x \in \Omega, t > 0.$$

A Banach space $\mathbb{B}$ which is not isomorphic to a Hilbert space?

- *Hytönen*, Rev. Mat. Iberoamericana, 2007: subordinated symetric difussion semigroups, *g*-functions
- *Kaiser and Weis*, Stud. Math., 2008: square functions of convolution type
- [HNP]: *Hytönen, Neerven and Portal*, J. d'Analyse Math., 2008: conical square functions

UMD Banach spaces $+$ γ-radonifying operators

Unconditionality of Martingale Differences (UMD)

Let $\mathfrak{H}$ be the Hilbert transform defined on $L^p(\mathbb{R}^n)$.
We say that a Banach space $\mathbb{B}$ is UMD when, for some (equivalently, all) $p \in (1,\infty)$, the Hilbert transform $\mathfrak{H} \otimes I_{\mathbb{B}}$ defined on $L^p(\mathbb{R}^n) \otimes \mathbb{B}$ can be extended as a bounded operator from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself.

- *Burkholder*, Ann. Prob., 1981; *Bourgain*, Ark. Mat., 1983 and *Rubio de Francia*, Lecture Notes in Math., 1986

A Banach space $\mathbb{B}$ which is not isomorphic to a Hilbert space?

- *Hytönen*, Rev. Mat. Iberoamericana, 2007: subordinated symetric difussion semigroups, g-functions
- *Kaiser and Weis*, Stud. Math., 2008: square functions of convolution type
- [HNP]: *Hytönen, Neerven and Portal*, J. d'Analyse Math., 2008: conical square functions

UMD Banach spaces $+$ γ-radonifying operators

Unconditionality of Martingale Differences (UMD)

Let $\mathfrak{H}$ be the Hilbert transform defined on $L^p(\mathbb{R}^n)$.
We say that a Banach space $\mathbb{B}$ is UMD when, for some (equivalently, all) $p \in (1, \infty)$, the Hilbert transform $\mathfrak{H} \otimes I_{\mathbb{B}}$ defined on $L^p(\mathbb{R}^n) \otimes \mathbb{B}$ can be extended as a bounded operator from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself.

- *Burkholder*, Ann. Prob., 1981; *Bourgain*, Ark. Mat., 1983 and *Rubio de Francia*, Lecture Notes in Math., 1986

A Banach space $\mathbb{B}$ which is not isomorphic to a Hilbert space?

- *Hytönen*, Rev. Mat. Iberoamericana, 2007: subordinated symetric difussion semigroups, *g*-functions
- *Kaiser and Weis*, Stud. Math., 2008: square functions of convolution type
- [HNP]: *Hytönen, Neerven and Portal*, J. d'Analyse Math., 2008: conical square functions

UMD Banach spaces $+$ γ-radonifying operators

Unconditionality of Martingale Differences (UMD)

Let $\mathfrak{H}$ be the Hilbert transform defined on $L^p(\mathbb{R}^n)$.
We say that a Banach space $\mathbb{B}$ is UMD when, for some (equivalently, all) $p \in (1,\infty)$, the Hilbert transform $\mathfrak{H} \otimes I_{\mathbb{B}}$ defined on $L^p(\mathbb{R}^n) \otimes \mathbb{B}$ can be extended as a bounded operator from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself.

- *Burkholder*, Ann. Prob., 1981; *Bourgain*, Ark. Mat., 1983 and *Rubio de Francia*, Lecture Notes in Math., 1986

γ-summing and γ-radonifying operators

Let H a Hilbert space and $\mathbb{B}$ a Banach space.
Assume that $\{\gamma_j\}_{j\in\mathbb{N}}$ is a sequence of independent standard Gaussian variables on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- $T \in \mathscr{L}(H,\mathbb{B})$ is γ-summing ($T \in \gamma^\infty(H,\mathbb{B})$) when

$$\|T\|_{\gamma^\infty(H,\mathbb{B})} := \sup \left(\mathbb{E}\Big\| \sum_{j=1}^k \gamma_j T h_j \Big\|_{\mathbb{B}}^2\right)^{1/2} < \infty.$$

(sup: all finite orthonormal systems $\{h_j\}_{j=1}^k$ in H)

- $\gamma(H,\mathbb{B})$: the closure in $\gamma^\infty(H,\mathbb{B})$ of all finite rank operators.

- Hoffmann-Jorgensen and Kwapień, Stud. Math., 1974:
If $\mathbb{B}$ does not contain an isomorphic copy of c_0, then $\gamma(H,\mathbb{B}) = \gamma^\infty(H,\mathbb{B})$.
- If H is separable and $\{h_j\}_{j=1}^\infty$ is an orthonormal basis of H, then:
$T \in \gamma(H,\mathbb{B})$ if and only if $\sum_{j=1}^\infty \gamma_j T h_j$ converges in $L^2(\Omega,\mathbb{B})$. In this case,

$$\|T\|_{\gamma(H,\mathbb{B})} = \left(\mathbb{E}\Big\| \sum_{j=1}^\infty \gamma_j T h_j \Big\|_{\mathbb{B}}^2\right)^{1/2}$$

γ-summing and γ-radonifying operators

Let H a Hilbert space and $\mathbb{B}$ a Banach space.
Assume that $\{\gamma_j\}_{j\in\mathbb{N}}$ is a sequence of independent standard Gaussian variables on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- $T \in \mathscr{L}(H,\mathbb{B})$ is γ-summing ($T \in \gamma^\infty(H,\mathbb{B})$) when

$$\|T\|_{\gamma^\infty(H,\mathbb{B})} := \sup \left(\mathbb{E}\left\|\sum_{j=1}^{k} \gamma_j T h_j\right\|_{\mathbb{B}}^2\right)^{1/2} < \infty.$$

  (sup: all finite orthonormal systems $\{h_j\}_{j=1}^k$ in H)
- $\gamma(H,\mathbb{B})$: the closure in $\gamma^\infty(H,\mathbb{B})$ of all finite rank operators.

- Hoffmann-Jorgensen and Kwapień, Stud. Math., 1974:
  If $\mathbb{B}$ does not contain an isomorphic copy of c_0, then $\gamma(H,\mathbb{B}) = \gamma^\infty(H,\mathbb{B})$.
- If H is separable and $\{h_j\}_{j=1}^\infty$ is an orthonormal basis of H, then:
  $T \in \gamma(H,\mathbb{B})$ if and only if $\sum_{j=1}^\infty \gamma_j T h_j$ converges in $L^2(\Omega,\mathbb{B})$. In this case,

$$\|T\|_{\gamma(H,\mathbb{B})} = \left(\mathbb{E}\left\|\sum_{j=1}^{\infty} \gamma_j T h_j\right\|_{\mathbb{B}}^2\right)^{1/2}$$

Space $\gamma(M, \mu; \mathbb{B})$

Let $(M, \mathscr{M}, \mu)$ be a σ-finite measure space and $H = L^2(M, \mu)$.
Assume that $f : M \longrightarrow \mathbb{B}$ is a weakly L^2 function.
Then, there exists $T_f \in \mathscr{L}(H, \mathbb{B})$ such that, for every $L \in \mathbb{B}^*$,

$$\langle L, T_f(h)\rangle_{\mathbb{B}^*, \mathbb{B}} = \int_M \langle L, f(w)\rangle_{\mathbb{B}^*, \mathbb{B}} h(w) d\mu(w), \quad h \in H.$$

We say that $f \in \gamma(M, \mu; \mathbb{B})$ when $T_f \in \gamma(H, \mathbb{B})$.
In that case we will write

$$\|f\|_{\gamma(H, \mathbb{B})} = \|T_f\|_{\gamma(H, \mathbb{B})}.$$

Conical square functions for UMD Banach spaces

Consider $H = L^2\left(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}\right)$.

Banach-valued tent spaces ([HNP])

Let $1 \leq p < \infty$. The space $T_p^2(\mathbb{R}^n, \mathbb{B})$ is defined as the completion of $C_c^\infty(\mathbb{R}^n \times (0,\infty)) \otimes \mathbb{B}$ with respect to the norm

$$\|f\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} := \|J(f)\|_{L^p(\mathbb{R}^n,\gamma(H,\mathbb{B}))},$$

where $[J(f)(x)](y,t) := \chi_{\Gamma(x)}(y,t) f(y,t), \quad x, y \in \mathbb{R}^n, t > 0.$

- $T_p^2(\mathbb{R}^n, \mathbb{C}) = T_p^2(\mathbb{R}^n)$ (the tent space of Coifman, Meyer and Stein, J.Funct. Anal., 1985)

Conical square functions for UMD Banach spaces

Consider $H = L^2\left(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}\right)$.

Banach-valued tent spaces ([HNP])

Let $1 \leq p < \infty$. The space $T_p^2(\mathbb{R}^n, \mathbb{B})$ is defined as the completion of $C_c^\infty(\mathbb{R}^n \times (0,\infty)) \otimes \mathbb{B}$ with respect to the norm

$$\|f\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} := \|J(f)\|_{L^p(\mathbb{R}^n,\gamma(H,\mathbb{B}))},$$

where $[J(f)(x)](y,t) := \chi_{\Gamma(x)}(y,t) f(y,t), \quad x, y \in \mathbb{R}^n, t > 0$.

- $T_p^2(\mathbb{R}^n, \mathbb{C}) = T_p^2(\mathbb{R}^n)$ (the tent space of Coifman, Meyer and Stein, J.Funct. Anal., 1985)

Scalar case (Torrea and Zhang, Proc. Roy. Soc. Edin., 2014)

Let $1 < p < \infty$ and $\beta > 0$. Then,

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

$\mathbb{B}$-valued case ([HNP])

Let B a UMD space and $1 < p < \infty$. For every $k \in \mathbb{N}$,

$$\|t^k \partial_t^k P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

Theorem 1

Let $\mathbb{B}$ a UMD space, $1 < p < \infty$ and $\beta > 0$. Then,

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

Scalar case (Torrea and Zhang, Proc. Roy. Soc. Edin., 2014)

Let $1 < p < \infty$ and $\beta > 0$. Then,

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

$\mathbb{B}$-valued case ([HNP])

Let B a UMD space and $1 < p < \infty$. For every $k \in \mathbb{N}$,

$$\|t^k \partial_t^k P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

Theorem 1

Let $\mathbb{B}$ a UMD space, $1 < p < \infty$ and $\beta > 0$. Then,

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

Scalar case (Torrea and Zhang, Proc. Roy. Soc. Edin., 2014)

Let $1 < p < \infty$ and $\beta > 0$. Then,

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

$\mathbb{B}$-valued case ([HNP])

Let B a UMD space and $1 < p < \infty$. For every $k \in \mathbb{N}$,

$$\|t^k \partial_t^k P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

Theorem 1

Let $\mathbb{B}$ a UMD space, $1 < p < \infty$ and $\beta > 0$. Then,

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

Ingredients of the proof

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$$

- $S(f)(y,t) := t^\beta \partial_t^\beta P_t(f)(y) = \int_{\mathbb{R}^n} k(y,z,t) f(z)dz, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$

$$|k(y,z,t)| := |t^\beta \partial_t^\beta P_t(y,z)| \leq C \frac{t^\beta}{(t+|y-z|)^{n+\beta}}, \quad y,z \in \mathbb{R}^n, t>0$$

- $\mathbb{S}(g)(y,t) := \int_{\mathbb{R}^n} k(y,z,t) g(z)dz, \quad g \in L^2(\mathbb{R}^n).$
  - $\|\mathbb{S}(g)\|_{T_2^2(\mathbb{R}^n)} = c_\beta \|g\|_{L^2(\mathbb{R}^n)}.$
  - $|k(y,z,t) - k(y,z',t)| \leq C \frac{t^\beta |z-z'|}{(t+|y-z|)^{n+\beta+1}}, \quad t+|y-z| > 2|z-z'|.$
  - $\int_{\mathbb{R}^n} k(y,z,t)dz = t^\beta \partial_t^\beta P_t(1)(y) = 0.$

[HNP]: $\mathbb{S}_{\mathbb{B}} = \mathbb{S} \otimes I_{\mathbb{B}}$ can be extended from $C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ to $L^p(\mathbb{R}^n,\mathbb{B})$ as a bounded operator $\widetilde{\mathbb{S}_{\mathbb{B}}}$ from $L^p(\mathbb{R}^n,\mathbb{B})$ into $T_p^2(\mathbb{R}^n,\mathbb{B})$.

Ingredients of the proof

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$$

- $S(f)(y,t) := t^\beta \partial_t^\beta P_t(f)(y) = \int_{\mathbb{R}^n} k(y,z,t)f(z)dz, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$

$$|k(y,z,t)| := |t^\beta \partial_t^\beta P_t(y,z)| \leq C\frac{t^\beta}{(t+|y-z|)^{n+\beta}}, \quad y,z \in \mathbb{R}^n, t > 0$$

- $\mathbb{S}(g)(y,t) := \int_{\mathbb{R}^n} k(y,z,t)g(z)dz, \quad g \in L^2(\mathbb{R}^n).$
  - $\|\mathbb{S}(g)\|_{T_2^2(\mathbb{R}^n)} = c_\beta \|g\|_{L^2(\mathbb{R}^n)}$.
  - $|k(y,z,t) - k(y,z',t)| \leq C\frac{t^\beta|z-z'|}{(t+|y-z|)^{n+\beta+1}}, \quad t+|y-z| > 2|z-z'|.$
  - $\int_{\mathbb{R}^n} k(y,z,t)dz = t^\beta \partial_t^\beta P_t(1)(y) = 0.$

[HNP]: $\mathbb{S}_\mathbb{B} = \mathbb{S} \otimes I_\mathbb{B}$ can be extended from $C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ to $L^p(\mathbb{R}^n,\mathbb{B})$ as a bounded operator $\widetilde{\mathbb{S}_\mathbb{B}}$ from $L^p(\mathbb{R}^n,\mathbb{B})$ into $T_p^2(\mathbb{R}^n,\mathbb{B})$.

Ingredients of the proof

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$$

- $S(f)(y,t) := t^\beta \partial_t^\beta P_t(f)(y) = \int_{\mathbb{R}^n} k(y,z,t)f(z)dz, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$

$$|k(y,z,t)| := |t^\beta \partial_t^\beta P_t(y,z)| \leq C\frac{t^\beta}{(t+|y-z|)^{n+\beta}}, \quad y,z \in \mathbb{R}^n, t>0$$

- $\mathbb{S}(g)(y,t) := \int_{\mathbb{R}^n} k(y,z,t)g(z)dz, \quad g \in L^2(\mathbb{R}^n).$
  - $\|\mathbb{S}(g)\|_{T_2^2(\mathbb{R}^n)} = c_\beta \|g\|_{L^2(\mathbb{R}^n)}.$
  - $|k(y,z,t)-k(y,z',t)| \leq C\frac{t^\beta|z-z'|}{(t+|y-z|)^{n+\beta+1}}, \quad t+|y-z|>2|z-z'|.$
  - $\int_{\mathbb{R}^n} k(y,z,t)dz = t^\beta\partial_t^\beta P_t(1)(y) = 0.$

[HNP]: $\mathbb{S}_\mathbb{B} = \mathbb{S}\otimes I_\mathbb{B}$ can be extended from $C_c^\infty(\mathbb{R}^n)\otimes\mathbb{B}$ to $L^p(\mathbb{R}^n,\mathbb{B})$ as a bounded operator $\widetilde{\mathbb{S}_\mathbb{B}}$ from $L^p(\mathbb{R}^n,\mathbb{B})$ into $T_p^2(\mathbb{R}^n,\mathbb{B})$.

Ingredients of the proof

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$$

- $S(f)(y,t) := t^\beta \partial_t^\beta P_t(f)(y) = \int_{\mathbb{R}^n} k(y,z,t)f(z)dz, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$

$$|k(y,z,t)| := |t^\beta \partial_t^\beta P_t(y,z)| \leq C\frac{t^\beta}{(t+|y-z|)^{n+\beta}}, \quad y,z \in \mathbb{R}^n, t>0$$

- $\mathbb{S}(g)(y,t) := \int_{\mathbb{R}^n} k(y,z,t)g(z)dz, \quad g \in L^2(\mathbb{R}^n).$
  - $\|\mathbb{S}(g)\|_{T_2^2(\mathbb{R}^n)} = c_\beta\|g\|_{L^2(\mathbb{R}^n)}.$
  - $|k(y,z,t)-k(y,z',t)| \leq C\frac{t^\beta|z-z'|}{(t+|y-z|)^{n+\beta+1}}, \quad t+|y-z|>2|z-z'|.$
  - $\int_{\mathbb{R}^n} k(y,z,t)dz = t^\beta \partial_t^\beta P_t(1)(y) = 0.$

[HNP]: $\mathbb{S}_{\mathbb{B}} = \mathbb{S}\otimes I_{\mathbb{B}}$ can be extended from $C_c^\infty(\mathbb{R}^n)\otimes\mathbb{B}$ to $L^p(\mathbb{R}^n,\mathbb{B})$ as a bounded operator $\widetilde{\mathbb{S}_{\mathbb{B}}}$ from $L^p(\mathbb{R}^n,\mathbb{B})$ into $T_p^2(\mathbb{R}^n,\mathbb{B})$.

Ingredients of the proof

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$$

- $S(f)(y,t) := t^\beta \partial_t^\beta P_t(f)(y) = \int_{\mathbb{R}^n} k(y,z,t) f(z)dz, \quad f \in L^p(\mathbb{R}^n,\mathbb{B})$.

$$|k(y,z,t)| := |t^\beta \partial_t^\beta P_t(y,z)| \leq C \frac{t^\beta}{(t+|y-z|)^{n+\beta}}, \quad y,z \in \mathbb{R}^n, t>0$$

- $\mathbb{S}(g)(y,t) := \int_{\mathbb{R}^n} k(y,z,t) g(z)dz, \quad g \in L^2(\mathbb{R}^n)$.
  - $\|\mathbb{S}(g)\|_{T_2^2(\mathbb{R}^n)} = c_\beta \|g\|_{L^2(\mathbb{R}^n)}$.
  - $|k(y,z,t) - k(y,z',t)| \leq C \frac{t^\beta |z-z'|}{(t+|y-z|)^{n+\beta+1}}, \quad t+|y-z| > 2|z-z'|$.
  - $\int_{\mathbb{R}^n} k(y,z,t)dz = t^\beta \partial_t^\beta P_t(1)(y) = 0$.

[HNP]: $\mathbb{S}_\mathbb{B} = \mathbb{S} \otimes I_\mathbb{B}$ can be extended from $C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ to $L^p(\mathbb{R}^n,\mathbb{B})$ as a bounded operator $\widetilde{\mathbb{S}_\mathbb{B}}$ from $L^p(\mathbb{R}^n,\mathbb{B})$ into $T_p^2(\mathbb{R}^n,\mathbb{B})$.

Ingredients of the proof

$$\|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$$

- $S(f)(y,t) := t^\beta \partial_t^\beta P_t(f)(y) = \int_{\mathbb{R}^n} k(y,z,t) f(z)dz, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$

$$|k(y,z,t)| := |t^\beta \partial_t^\beta P_t(y,z)| \leq C\frac{t^\beta}{(t+|y-z|)^{n+\beta}}, \quad y,z \in \mathbb{R}^n, t>0$$

- $\mathbb{S}(g)(y,t) := \int_{\mathbb{R}^n} k(y,z,t) g(z)dz, \quad g \in L^2(\mathbb{R}^n).$
  - $\|\mathbb{S}(g)\|_{T_2^2(\mathbb{R}^n)} = c_\beta \|g\|_{L^2(\mathbb{R}^n)}.$
  - $|k(y,z,t) - k(y,z',t)| \leq C\frac{t^\beta |z-z'|}{(t+|y-z|)^{n+\beta+1}}, \quad t+|y-z| > 2|z-z'|.$
  - $\int_{\mathbb{R}^n} k(y,z,t)dz = t^\beta \partial_t^\beta P_t(1)(y) = 0.$

[HNP]: $\mathbb{S}_{\mathbb{B}} = \mathbb{S} \otimes I_{\mathbb{B}}$ can be extended from $C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ to $L^p(\mathbb{R}^n,\mathbb{B})$ as a bounded operator $\widetilde{\mathbb{S}_{\mathbb{B}}}$ from $L^p(\mathbb{R}^n,\mathbb{B})$ into $T_p^2(\mathbb{R}^n,\mathbb{B})$.

$$S = \widetilde{\mathbb{S}_{\mathbb{B}}} \quad \text{on } L^p(\mathbb{R}^n, \mathbb{B})$$

Let $f \in L^p(\mathbb{R}^n, \mathbb{B})$ and $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ such that $f_k \longrightarrow f$, in $L^p(\mathbb{R}^n, \mathbb{B})$. Then,

$$\widetilde{\mathbb{S}_{\mathbb{B}}} f_k = \mathbb{S}_{\mathbb{B}} f_k \longrightarrow \widetilde{\mathbb{S}_{\mathbb{B}}} f, \quad \text{in } T_p^2(\mathbb{R}^n, \mathbb{B}).$$

Then there exists $G \in L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, $\Omega \subset \mathbb{R}^n : |\mathbb{R}^n \setminus \Omega| = 0$ and $(k_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{N}$:

$$J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow G(x), \quad x \in \Omega, \text{ in } \gamma(H, \mathbb{B}).$$

Since $\gamma(H, \mathbb{B}) \subset \mathscr{L}(H, \mathbb{B})$, it follows that, for every $L \in \mathbb{B}^*$,

$$\langle L, \langle J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x), h\rangle\rangle \longrightarrow \langle L, G(x)(h)\rangle, \quad h \in H.$$

- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \in L^2(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B}), \quad x \in W_\ell, \ |\mathbb{R}^n \setminus W_\ell| = 0, \quad \ell \in \mathbb{N}.$
- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow J(Sf)(x) \in L^2(\mathbb{R}^n \times (\varepsilon,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B}), \quad \varepsilon > 0.$

$$S = \widetilde{\mathbb{S}_{\mathbb{B}}} \quad \text{on } L^p(\mathbb{R}^n, \mathbb{B})$$

Let $f \in L^p(\mathbb{R}^n, \mathbb{B})$ and $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ such that $f_k \longrightarrow f$, in $L^p(\mathbb{R}^n, \mathbb{B})$.

Then,

$$\widetilde{\mathbb{S}_{\mathbb{B}}} f_k = \mathbb{S}_{\mathbb{B}} f_k \longrightarrow \widetilde{\mathbb{S}_{\mathbb{B}}} f, \quad \text{in } T_p^2(\mathbb{R}^n, \mathbb{B}).$$

Then there exists $G \in L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, $\Omega \subset \mathbb{R}^n : |\mathbb{R}^n \setminus \Omega| = 0$ and $(k_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{N}$:

$$J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow G(x), \quad x \in \Omega, \text{ in } \gamma(H, \mathbb{B}).$$

Since $\gamma(H, \mathbb{B}) \subset \mathscr{L}(H, \mathbb{B})$, it follows that, for every $L \in \mathbb{B}^*$,

$$\langle L, \langle J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x), h\rangle\rangle \longrightarrow \langle L, G(x)(h)\rangle, \quad h \in H.$$

- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \in L^2(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B}), \quad x \in W_\ell, \ |\mathbb{R}^n \setminus W_\ell| = 0, \quad \ell \in \mathbb{N}.$
- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow J(Sf)(x) \in L^2(\mathbb{R}^n \times (\varepsilon,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B}), \quad \varepsilon > 0.$

$$S = \widetilde{\mathbb{S}_{\mathbb{B}}} \quad \text{on } L^p(\mathbb{R}^n, \mathbb{B})$$

Let $f \in L^p(\mathbb{R}^n, \mathbb{B})$ and $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ such that $f_k \longrightarrow f$, in $L^p(\mathbb{R}^n, \mathbb{B})$. Then,

$$\widetilde{\mathbb{S}_{\mathbb{B}}} f_k = \mathbb{S}_{\mathbb{B}} f_k \longrightarrow \widetilde{\mathbb{S}_{\mathbb{B}}} f, \quad \text{in } T_p^2(\mathbb{R}^n, \mathbb{B}).$$

Then there exists $G \in L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, $\Omega \subset \mathbb{R}^n : |\mathbb{R}^n \setminus \Omega| = 0$ and $(k_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{N}$:

$$J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow G(x), \quad x \in \Omega, \text{ in } \gamma(H, \mathbb{B}).$$

Since $\gamma(H, \mathbb{B}) \subset \mathscr{L}(H, \mathbb{B})$, it follows that, for every $L \in \mathbb{B}^*$,

$$\langle L, \langle J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x), h\rangle\rangle \longrightarrow \langle L, G(x)(h)\rangle, \quad h \in H.$$

- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \in L^2(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B}), \quad x \in W_\ell, \; |\mathbb{R}^n \setminus W_\ell| = 0, \quad \ell \in \mathbb{N}.$
- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow J(Sf)(x) \in L^2(\mathbb{R}^n \times (\varepsilon,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B}), \quad \varepsilon > 0.$

$$S = \widetilde{\mathbb{S}_{\mathbb{B}}} \quad \text{on } L^p(\mathbb{R}^n, \mathbb{B})$$

Let $f \in L^p(\mathbb{R}^n, \mathbb{B})$ and $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ such that $f_k \longrightarrow f$, in $L^p(\mathbb{R}^n, \mathbb{B})$. Then,

$$\widetilde{\mathbb{S}_{\mathbb{B}}} f_k = \mathbb{S}_{\mathbb{B}} f_k \longrightarrow \widetilde{\mathbb{S}_{\mathbb{B}}} f, \quad \text{in } T_p^2(\mathbb{R}^n, \mathbb{B}).$$

Then there exists $G \in L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, $\Omega \subset \mathbb{R}^n : |\mathbb{R}^n \setminus \Omega| = 0$ and $(k_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{N}$:

$$J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow G(x), \quad x \in \Omega, \text{ in } \gamma(H, \mathbb{B}).$$

Since $\gamma(H, \mathbb{B}) \subset \mathscr{L}(H, \mathbb{B})$, it follows that, for every $L \in \mathbb{B}^*$,

$$\langle L, \langle J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x), h\rangle\rangle \longrightarrow \langle L, G(x)(h)\rangle, \quad h \in H.$$

- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \in L^2(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B})$, $\quad x \in W_\ell$, $|\mathbb{R}^n \setminus W_\ell| = 0$, $\quad \ell \in \mathbb{N}$.
- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow J(Sf)(x) \in L^2(\mathbb{R}^n \times (\varepsilon,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B})$, $\quad \varepsilon > 0$.

$$S = \widetilde{\mathbb{S}_{\mathbb{B}}} \quad \text{on } L^p(\mathbb{R}^n, \mathbb{B})$$

Let $f \in L^p(\mathbb{R}^n, \mathbb{B})$ and $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ such that $f_k \longrightarrow f$, in $L^p(\mathbb{R}^n, \mathbb{B})$. Then,

$$\widetilde{\mathbb{S}_{\mathbb{B}}} f_k = \mathbb{S}_{\mathbb{B}} f_k \longrightarrow \widetilde{\mathbb{S}_{\mathbb{B}}} f, \quad \text{in } T_p^2(\mathbb{R}^n, \mathbb{B}).$$

Then there exists $G \in L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, $\Omega \subset \mathbb{R}^n : |\mathbb{R}^n \setminus \Omega| = 0$ and $(k_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{N}$:

$$J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow G(x), \quad x \in \Omega, \text{ in } \gamma(H, \mathbb{B}).$$

Since $\gamma(H, \mathbb{B}) \subset \mathscr{L}(H, \mathbb{B})$, it follows that, for every $L \in \mathbb{B}^*$,

$$\langle L, \langle J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x), h\rangle\rangle \longrightarrow \langle L, G(x)(h)\rangle, \quad h \in H.$$

- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \in L^2(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B}), \quad x \in W_\ell, \ |\mathbb{R}^n \setminus W_\ell| = 0, \quad \ell \in \mathbb{N}.$
- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow J(Sf)(x) \in L^2(\mathbb{R}^n \times (\varepsilon,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B}), \quad \varepsilon > 0.$

$$S = \widetilde{\mathbb{S}_{\mathbb{B}}} \quad \text{on } L^p(\mathbb{R}^n, \mathbb{B})$$

Let $f \in L^p(\mathbb{R}^n, \mathbb{B})$ and $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ such that $f_k \longrightarrow f$, in $L^p(\mathbb{R}^n, \mathbb{B})$. Then,

$$\widetilde{\mathbb{S}_{\mathbb{B}}} f_k = \mathbb{S}_{\mathbb{B}} f_k \longrightarrow \widetilde{\mathbb{S}_{\mathbb{B}}} f, \quad \text{in } T_p^2(\mathbb{R}^n, \mathbb{B}).$$

Then there exists $G \in L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, $\Omega \subset \mathbb{R}^n : |\mathbb{R}^n \setminus \Omega| = 0$ and $(k_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{N}$:

$$J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow G(x), \quad x \in \Omega, \text{ in } \gamma(H, \mathbb{B}).$$

Since $\gamma(H, \mathbb{B}) \subset \mathscr{L}(H, \mathbb{B})$, it follows that, for every $L \in \mathbb{B}^*$,

$$\langle L, \langle J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x), h \rangle\rangle \longrightarrow \langle L, G(x)(h)\rangle, \quad h \in H.$$

- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \in L^2(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B})$, $\quad x \in W_\ell$, $|\mathbb{R}^n \setminus W_\ell| = 0$, $\quad \ell \in \mathbb{N}$.
- $J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x) \longrightarrow J(Sf)(x) \in L^2(\mathbb{R}^n \times (\varepsilon,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B})$, $\quad \varepsilon > 0$.

We obtain that, there exits $\Omega_0 \subset \mathbb{R}^n$, with $|\mathbb{R}^n \setminus \Omega_0| = 0$ such that

$$\int_{\mathbb{R}^n \times (0,\infty)} \langle L, [J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x)](y,t)\rangle h(y,t) \frac{dydt}{t^{n+1}} \longrightarrow \langle L, G(x)(h)\rangle, \quad x \in \Omega_0,\ h \in H.$$

$$\downarrow$$

$$\int_{\mathbb{R}^n \times (0,\infty)} \langle L, [J(Sf)(x)](y,t)\rangle h(y,t) \frac{dydt}{t^{n+1}}, \quad x \in \mathbb{R}^n,\ h \in C_c^\infty(\mathbb{R}^n \times (0,\infty)).$$

From here we can see that, for $x \in \Omega_0$, $[J(Sf)(x)](\cdot,\cdot)$ is weakly L^2 and then

$$J(Sf)(x) = G(x), \quad x \in \Omega_0, \text{ as elements of } \gamma(H,\mathbb{B}).$$

We obtain that, there exits $\Omega_0 \subset \mathbb{R}^n$, with $|\mathbb{R}^n \setminus \Omega_0| = 0$ such that

$$\int_{\mathbb{R}^n\times(0,\infty)} \langle L, [J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x)](y,t)\rangle h(y,t)\,\frac{dydt}{t^{n+1}} \longrightarrow \langle L, G(x)(h)\rangle, \quad x \in \Omega_0,\ h \in H.$$

$$\downarrow$$

$$\int_{\mathbb{R}^n\times(0,\infty)} \langle L, [J(Sf)(x)](y,t)\rangle h(y,t)\,\frac{dydt}{t^{n+1}}, \quad x \in \mathbb{R}^n,\ h \in C_c^\infty(\mathbb{R}^n \times (0,\infty)).$$

From here we can see that, for $x \in \Omega_0$, $[J(Sf)(x)](\cdot,\cdot)$ is weakly L^2 and then

$$J(Sf)(x) = G(x), \quad x \in \Omega_0, \text{ as elements of } \gamma(H,\mathbb{B}).$$

We obtain that, there exits $\Omega_0 \subset \mathbb{R}^n$, with $|\mathbb{R}^n \setminus \Omega_0| = 0$ such that

$$\int_{\mathbb{R}^n\times(0,\infty)} \langle L, [J(\mathbb{S}_{\mathbb{B}} f_{k_\ell})(x)](y,t)\rangle h(y,t)\frac{dydt}{t^{n+1}} \longrightarrow \langle L, G(x)(h)\rangle, \quad x \in \Omega_0,\ h \in H.$$

$$\downarrow$$

$$\int_{\mathbb{R}^n\times(0,\infty)} \langle L, [J(Sf)(x)](y,t)\rangle h(y,t)\frac{dydt}{t^{n+1}}, \quad x \in \mathbb{R}^n,\ h \in C_c^\infty(\mathbb{R}^n \times (0,\infty)).$$

From here we can see that, for $x \in \Omega_0$, $[J(Sf)(x)](\cdot,\cdot)$ is weakly L^2 and then

$$J(Sf)(x) = G(x), \quad x \in \Omega_0, \text{ as elements of } \gamma(H,\mathbb{B}).$$

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq \|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})}$$

- Polarization formula: for $f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ and $g \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}^*$,

$$\int_{\mathbb{R}^n} \int_{\Gamma(x)} \langle t^\beta \partial_t^\beta P_t(g)(y), t^\beta \partial_t^\beta P_t(f)(y)\rangle \frac{dydt}{t^{n+1}} dx = c_{n,\beta} \int_{\mathbb{R}^n} \langle g(y), f(y)\rangle dy$$

$$\left(t^\beta \partial_t^\beta P_t(F)(y) = e^{i\pi\beta}(|tz|^\beta e^{-t|z|}\widehat{F})^\vee \right)$$

- Kalton and Weis's result: Let $f \in \gamma(M,\mu;\mathbb{B})$ and $g \in \gamma(M,\mu;\mathbb{B}^*)$ strongly measurable functions on (M,μ). Then,

$$\int_M |\langle g(w), f(w)\rangle| d\mu(w) \leq \|f\|_{\gamma(H,\mathbb{B})} \|g\|_{\gamma(H,\mathbb{B}^*)}$$

- Duality and density arguments.

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq \|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})}$$

- Polarization formula: for $f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ and $g \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}^*$,

$$\int_{\mathbb{R}^n} \int_{\Gamma(x)} \langle t^\beta \partial_t^\beta P_t(g)(y), t^\beta \partial_t^\beta P_t(f)(y) \rangle \frac{dydt}{t^{n+1}} dx = c_{n,\beta} \int_{\mathbb{R}^n} \langle g(y), f(y) \rangle dy$$

$$\left(t^\beta \partial_t^\beta P_t(F)(y) = e^{i\pi\beta} (|tz|^\beta e^{-t|z|} \widehat{F})^\vee \right)$$

- Kalton and Weis's result: Let $f \in \gamma(M,\mu;\mathbb{B})$ and $g \in \gamma(M,\mu;\mathbb{B}^*)$ strongly measurable functions on (M,μ). Then,

$$\int_M |\langle g(w), f(w) \rangle| d\mu(w) \leq \|f\|_{\gamma(H,\mathbb{B})} \|g\|_{\gamma(H,\mathbb{B}^*)}$$

- Duality and density arguments.

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq \|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})}$$

- Polarization formula: for $f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ and $g \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}^*$,

$$\int_{\mathbb{R}^n} \int_{\Gamma(x)} \langle t^\beta \partial_t^\beta P_t(g)(y), t^\beta \partial_t^\beta P_t(f)(y) \rangle \frac{dydt}{t^{n+1}} dx = c_{n,\beta} \int_{\mathbb{R}^n} \langle g(y), f(y) \rangle dy$$

$$\left(t^\beta \partial_t^\beta P_t(F)(y) = e^{i\pi\beta} (|tz|^\beta e^{-t|z|} \widehat{F})^\vee \right)$$

- Kalton and Weis's result: Let $f \in \gamma(M,\mu;\mathbb{B})$ and $g \in \gamma(M,\mu;\mathbb{B}^*)$ strongly measurable functions on (M,μ). Then,

$$\int_M |\langle g(w), f(w) \rangle| d\mu(w) \leq \|f\|_{\gamma(H,\mathbb{B})} \|g\|_{\gamma(H,\mathbb{B}^*)}$$

- Duality and density arguments.

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq \|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})}$$

- Polarization formula: for $f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ and $g \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}^*$,

$$\int_{\mathbb{R}^n} \int_{\Gamma(x)} \langle t^\beta \partial_t^\beta P_t(g)(y), t^\beta \partial_t^\beta P_t(f)(y)\rangle \frac{dydt}{t^{n+1}} dx = c_{n,\beta} \int_{\mathbb{R}^n} \langle g(y), f(y)\rangle dy$$

$$\left(t^\beta \partial_t^\beta P_t(F)(y) = e^{i\pi\beta}(|tz|^\beta e^{-t|z|}\widehat{F})^\vee\right)$$

- Kalton and Weis's result: Let $f \in \gamma(M,\mu;\mathbb{B})$ and $g \in \gamma(M,\mu;\mathbb{B}^*)$ strongly measurable functions on (M,μ). Then,

$$\int_M |\langle g(w), f(w)\rangle| d\mu(w) \leq \|f\|_{\gamma(H,\mathbb{B})} \|g\|_{\gamma(H,\mathbb{B}^*)}$$

- Duality and density arguments.

1. Square functions associated to the Laplacian

2. Conical square functions associated to other operators
   - Schrödinger operator. Harmonic oscillator
   - Bessel and Laguerre operator

Schrödinger context

Let us consider $V \not\equiv 0$, $V \geq 0$, $V \in RH_s$ for some $s > n/2$:

$$\Big(\int_B V(x)^s dx\Big)^{1/s} \leq C \int_B V(x)dx, \quad B \text{ ball in } \mathbb{R}^n.$$

Schrödinger operator: $L_V = -\Delta + V$ in $\mathbb{R}^n$.

We define $\mathscr{L}_V$ as the unique selfadjoint operator with domain

$$\mathscr{D}_V = \{f \in L^2(\mathbb{R}^n) : |\nabla f| \in L^2(\mathbb{R}^n) \text{ and } \sqrt{V} f \in L^2(\mathbb{R}^n)\}$$

and such that

$$\langle \mathscr{L}_V f, g\rangle = \int_{\mathbb{R}^n} \langle \nabla f(x), \overline{\nabla g(x)}\rangle dx + \int_{\mathbb{R}^n} V(x) f(x) \overline{g(x)} dx, \quad f, g \in \mathscr{D}_V.$$

- $\mathscr{L}_V$ is a positive operator and $\mathscr{L}_V = L_V$ on $C_c^\infty(\mathbb{R}^n)$.

Schrödinger context

Let us consider $V \not\equiv 0$, $V \geq 0$, $V \in RH_s$ for some $s > n/2$:

$$\Big(\int_B V(x)^s dx\Big)^{1/s} \leq C \int_B V(x)dx, \quad B \text{ ball in } \mathbb{R}^n.$$

Schrödinger operator: $L_V = -\Delta + V$ in $\mathbb{R}^n$.

We define $\mathscr{L}_V$ as the unique selfadjoint operator with domain

$$\mathscr{D}_V = \{f \in L^2(\mathbb{R}^n) : |\nabla f| \in L^2(\mathbb{R}^n) \text{ and } \sqrt{V} f \in L^2(\mathbb{R}^n)\}$$

and such that

$$\langle \mathscr{L}_V f, g\rangle = \int_{\mathbb{R}^n} \langle \nabla f(x), \overline{\nabla g(x)}\rangle dx + \int_{\mathbb{R}^n} V(x) f(x) \overline{g(x)} dx, \quad f, g \in \mathscr{D}_V.$$

- $\mathscr{L}_V$ is a positive operator and $\mathscr{L}_V = L_V$ on $C_c^\infty(\mathbb{R}^n)$.

Schrödinger context

Let us consider $V \not\equiv 0$, $V \geq 0$, $V \in RH_s$ for some $s > n/2$:

$$\left(\int_B V(x)^s dx\right)^{1/s} \leq C \int_B V(x)dx, \quad B \text{ ball in } \mathbb{R}^n.$$

Schrödinger operator: $L_V = -\Delta + V$ in $\mathbb{R}^n$.

We define $\mathscr{L}_V$ as the unique selfadjoint operator with domain

$$\mathscr{D}_V = \{f \in L^2(\mathbb{R}^n) : |\nabla f| \in L^2(\mathbb{R}^n) \text{ and } \sqrt{V} f \in L^2(\mathbb{R}^n)\}$$

and such that

$$\langle \mathscr{L}_V f, g\rangle = \int_{\mathbb{R}^n} \langle \nabla f(x), \overline{\nabla g(x)}\rangle dx + \int_{\mathbb{R}^n} V(x) f(x) \overline{g(x)} dx, \quad f, g \in \mathscr{D}_V.$$

- $\mathscr{L}_V$ is a positive operator and $\mathscr{L}_V = L_V$ on $C_c^\infty(\mathbb{R}^n)$.

Schrödinger context

Let us consider $V \not\equiv 0$, $V \geq 0$, $V \in RH_s$ for some $s > n/2$:

$$\left(\int_B V(x)^s dx \right)^{1/s} \leq C \int_B V(x) dx, \quad B \text{ ball in } \mathbb{R}^n.$$

Schrödinger operator: $L_V = -\Delta + V$ in $\mathbb{R}^n$.

We define $\mathscr{L}_V$ as the unique selfadjoint operator with domain

$$\mathscr{D}_V = \{ f \in L^2(\mathbb{R}^n) : |\nabla f| \in L^2(\mathbb{R}^n) \text{ and } \sqrt{V} f \in L^2(\mathbb{R}^n) \}$$

and such that

$$\langle \mathscr{L}_V f, g \rangle = \int_{\mathbb{R}^n} \langle \nabla f(x), \overline{\nabla g(x)} \rangle dx + \int_{\mathbb{R}^n} V(x) f(x) \overline{g(x)} dx, \quad f, g \in \mathscr{D}_V.$$

- $\mathscr{L}_V$ is a positive operator and $\mathscr{L}_V = L_V$ on $C_c^\infty(\mathbb{R}^n)$.

Schrödinger context

- Heat semigroup $\{W_t^{\mathscr{L}_V}\}_{t>0}$ generated by $-\mathscr{L}_V$ in $L^2(\mathbb{R}^n)$:

$$W_t^{\mathscr{L}_V}(f)(x) = \int_{\mathbb{R}^n} W_t^{\mathscr{L}_V}(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^n),$$

$$|W_t^{\mathscr{L}_V}(x,y)| \leq C\frac{e^{-|x-y|^2/(4t)}}{t^{n/2}}.$$

  - $\{W_t^{\mathscr{L}_V}\}_{t>0}$ is a positive bounded semigroup in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

- Poisson semigroup $\{P_t^{\mathscr{L}_V}\}_{t>0}$: by subordination

$$P_t^{\mathscr{L}_V}(f)(x) = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}} W_u^{\mathscr{L}_V}(f)(x)du, \quad f \in L^p(\mathbb{R}^n).$$

Schrödinger context

- Heat semigroup $\{W_t^{\mathscr{L}_V}\}_{t>0}$ generated by $-\mathscr{L}_V$ in $L^2(\mathbb{R}^n)$:

$$W_t^{\mathscr{L}_V}(f)(x) = \int_{\mathbb{R}^n} W_t^{\mathscr{L}_V}(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^n),$$

$$|W_t^{\mathscr{L}_V}(x,y)| \leq C\frac{e^{-|x-y|^2/(4t)}}{t^{n/2}}.$$

  - $\{W_t^{\mathscr{L}_V}\}_{t>0}$ is a positive bounded semigroup in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.
- Poisson semigroup $\{P_t^{\mathscr{L}_V}\}_{t>0}$: by subordination

$$P_t^{\mathscr{L}_V}(f)(x) = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}} W_u^{\mathscr{L}_V}(f)(x)du, \quad f \in L^p(\mathbb{R}^n).$$

Hermite context

$\mathscr{H}$: particular case of Schrödinger operator, when $V(x) = |x|^2$.

- Heat semigroup $\{W_t^{\mathscr{H}}\}_{t>0}$:

$$W_t^{\mathscr{H}}(f)(x) = \int_{\mathbb{R}^n} W_t^{\mathscr{H}}(x,y)f(y)dy, \quad f \in L^p(\mathbb{R}^n),$$

where

$$W_t^{\mathscr{H}}(x,y) = \left(\frac{e^{-2t}}{\pi(1-e^{-4t})}\right)^{n/2} \exp\left[-\frac{1}{4}\left(|x-y|^2\frac{1+e^{-2t}}{1-e^{-2t}} + |x+y|^2\frac{1-e^{-2t}}{1+e^{-2t}}\right)\right]$$

- Poisson semigroup $\{P_t^{\mathscr{H}}\}_{t>0}$: by subordination.

Hermite context

$\mathscr{H}$: particular case of Schrödinger operator, when $V(x) = |x|^2$.

- Heat semigroup $\{W_t^{\mathscr{H}}\}_{t>0}$:

$$W_t^{\mathscr{H}}(f)(x) = \int_{\mathbb{R}^n} W_t^{\mathscr{H}}(x,y)f(y)dy, \quad f \in L^p(\mathbb{R}^n),$$

  where

$$W_t^{\mathscr{H}}(x,y) = \left(\frac{e^{-2t}}{\pi(1-e^{-4t})}\right)^{n/2} \exp\left[-\frac{1}{4}\left(|x-y|^2\frac{1+e^{-2t}}{1-e^{-2t}} + |x+y|^2\frac{1-e^{-2t}}{1+e^{-2t}}\right)\right]$$

- Poisson semigroup $\{P_t^{\mathscr{H}}\}_{t>0}$: by subordination.

Theorem 2

Let $1 < p < \infty$ and $\beta > 0$. Assume that $\mathbb{B}$ is a UMD Banach space.

(i) If $V \in RH_s$ for some $s > n/2$ and $n \geq 3$, then

$$\|t^{\beta}\partial_t^{\beta} P_t^{\mathscr{L}_V}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

(ii) For every $n \in \mathbb{N}$,

$$\|t^{\beta}\partial_t^{\beta} P_t^{\mathscr{H}}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

Remark: Since $\partial_t P_t^{\mathscr{L}_V}(1) \not\equiv 0$ we can not apply the result on [HNP].

Theorem 2

Let $1 < p < \infty$ and $\beta > 0$. Assume that $\mathbb{B}$ is a UMD Banach space.

(i) If $V \in RH_s$ for some $s > n/2$ and $n \geq 3$, then

$$\|t^{\beta}\partial_t^{\beta} P_t^{\mathcal{L}_V}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

(ii) For every $n \in \mathbb{N}$,

$$\|t^{\beta}\partial_t^{\beta} P_t^{\mathcal{H}}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

Remark: Since $\partial_t P_t^{\mathcal{L}_V}(1) \not\equiv 0$ we can not apply the result on [HNP].

Scheme of the proof

$$\|t^\beta \partial_t^\beta P_t^{\mathscr{L}_V}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})},\ f \in C_c^\infty(\mathbb{R}^n)\otimes\mathbb{B}.$$

Let $K^{\mathscr{L}_V}$ and $\mathbb{K}$ be the operators given by

$$K^{\mathscr{L}_V}(f)(x;y,t) = \chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t^{\mathscr{L}_V}(f)(y) \quad \text{and} \quad \mathbb{K}(f)(x;y,t) = \chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t(f)(y).$$

Consider the critical radius $\rho(x)$, $x \in \mathbb{R}^n$, given by

$$\rho(x) = \sup\left\{r > 0 : r^{2-n}\int_{B(x,r)} V(y)dy \leq 1\right\}.$$

We write

$$K^{\mathscr{L}_V}(f) = K_{\text{glob}}^{\mathscr{L}_V}(f) + \underbrace{(K_{\text{loc}}^{\mathscr{L}_V}(f) - \mathbb{K}_{\text{loc}}(f))}_{D^{\mathscr{L}_V}(f)} + \mathbb{K}_{\text{loc}}(f)$$

where $T_{\text{loc}}(f)(x;\cdot,\cdot) = T(f\chi_{B(x,\rho(x))})(x;\cdot,\cdot)$ for $T = K^{\mathscr{L}_V}$ or $\mathbb{K}$.

Scheme of the proof

$$\|t^\beta \partial_t^\beta P_t^{\mathcal{L}_V}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \; f \in C_c^\infty(\mathbb{R}^n)\otimes\mathbb{B}.$$

Let $K^{\mathcal{L}_V}$ and $\mathbb{K}$ be the operators given by

$$K^{\mathcal{L}_V}(f)(x;y,t) = \chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t^{\mathcal{L}_V}(f)(y) \quad \text{and} \quad \mathbb{K}(f)(x;y,t) = \chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t(f)(y).$$

Consider the critical radius $\rho(x)$, $x \in \mathbb{R}^n$, given by

$$\rho(x) = \sup\left\{ r > 0 : r^{2-n}\int_{B(x,r)} V(y)dy \leq 1 \right\}.$$

We write

$$K^{\mathcal{L}_V}(f) = K_{\text{glob}}^{\mathcal{L}_V}(f) + \underbrace{(K_{\text{loc}}^{\mathcal{L}_V}(f) - \mathbb{K}_{\text{loc}}(f))}_{D^{\mathcal{L}_V}(f)} + \mathbb{K}_{\text{loc}}(f)$$

where $T_{\text{loc}}(f)(x;\cdot,\cdot) = T(f\chi_{B(x,\rho(x))})(x;\cdot,\cdot)$ for $T = K^{\mathcal{L}_V}$ or $\mathbb{K}$.

Scheme of the proof

$$\|t^\beta \partial_t^\beta P_t^{\mathcal{L}_V}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \; f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}.$$

Let $K^{\mathcal{L}_V}$ and $\mathbb{K}$ be the operators given by

$$K^{\mathcal{L}_V}(f)(x;y,t) = \chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t^{\mathcal{L}_V}(f)(y) \quad \text{and} \quad \mathbb{K}(f)(x;y,t) = \chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t(f)(y).$$

Consider the critical radius $\rho(x)$, $x \in \mathbb{R}^n$, given by

$$\rho(x) = \sup\left\{ r > 0 : r^{2-n} \int_{B(x,r)} V(y)dy \leq 1 \right\}.$$

We write

$$K^{\mathcal{L}_V}(f) = K_{\text{glob}}^{\mathcal{L}_V}(f) + \underbrace{(K_{\text{loc}}^{\mathcal{L}_V}(f) - \mathbb{K}_{\text{loc}}(f))}_{D^{\mathcal{L}_V}(f)} + \mathbb{K}_{\text{loc}}(f)$$

where $T_{\text{loc}}(f)(x;\cdot,\cdot) = T(f\chi_{B(x,\rho(x))})(x;\cdot,\cdot)$ for $T = K^{\mathcal{L}_V}$ or $\mathbb{K}$.

Scheme of the proof

$$\|t^\beta \partial_t^\beta P_t^{\mathcal{L}_V}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \; f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}.$$

Let $K^{\mathcal{L}_V}$ and $\mathbb{K}$ be the operators given by

$$K^{\mathcal{L}_V}(f)(x;y,t) = \chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t^{\mathcal{L}_V}(f)(y) \quad \text{and} \quad \mathbb{K}(f)(x;y,t) = \chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t(f)(y).$$

Consider the critical radius $\rho(x)$, $x \in \mathbb{R}^n$, given by

$$\rho(x) = \sup\left\{ r > 0 : r^{2-n} \int_{B(x,r)} V(y)dy \leq 1 \right\}.$$

We write

$$K^{\mathcal{L}_V}(f) = K_{\rm glob}^{\mathcal{L}_V}(f) + \underbrace{(K_{\rm loc}^{\mathcal{L}_V}(f) - \mathbb{K}_{\rm loc}(f))}_{D^{\mathcal{L}_V}(f)} + \mathbb{K}_{\rm loc}(f)$$

where $T_{\rm loc}(f)(x;\cdot,\cdot) = T(f\chi_{B(x,\rho(x))})(x;\cdot,\cdot)$ for $T = K^{\mathcal{L}_V}$ or $\mathbb{K}$.

It is established that:

(1) $\|K^{\mathcal{L}_\nu}_{\mathrm{glob}}(f)(x;\cdot,\cdot)\|_{\gamma(H,\mathbb{B})} \leq C\mathcal{M}(\|f\|_{\mathbb{B}})(x).$

(2) $\|D^{\mathcal{L}_\nu}(f)(x;\cdot,\cdot)\|_{\gamma(H,\mathbb{B})} \leq C(W_*(\|f\|_{\mathbb{B}})(x) + \mathcal{M}(\|f\|_{\mathbb{B}})(x))$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator and W_* the maximal operator associated to the classical heat semigroup.

Steps in the proof of this type of inequalities: Let $G(x) = K^{\mathcal{L}}_{\mathrm{glob}}(f)(x;\cdot,\cdot)$, $x \in \mathbb{R}^n$.

- $G(x) \in L^2(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}};\mathbb{B})$, $x \in \mathbb{R}^n$.
- $\|G(x)\|_{\gamma(H,\mathbb{B})} = \left(\mathbb{E}\left\|\sum_{j=1}^{\infty} \gamma_j T_{G(x)}(h_j)\right\|_{\mathbb{B}}^2\right)^{1/2}$

$$\leq C\int_{\mathbb{R}^n\setminus B(x,\rho(x))} \|f(z)\|_{\mathbb{B}} \|\chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t^{\mathcal{L}_\nu}(y,z)\|_H dz$$

$$\leq \cdots \leq C\mathcal{M}(\|f\|_{\mathbb{B}})(x), \quad x \in \mathbb{R}^n.$$

It is established that:

(1) $\|K^{\mathcal{L}_\nu}_{\rm glob}(f)(x;\cdot,\cdot)\|_{\gamma(H,\mathbb{B})} \leq C\mathscr{M}(\|f\|_{\mathbb{B}})(x)$.

(2) $\|D^{\mathcal{L}_\nu}(f)(x;\cdot,\cdot)\|_{\gamma(H,\mathbb{B})} \leq C(W_*(\|f\|_{\mathbb{B}})(x) + \mathscr{M}(\|f\|_{\mathbb{B}})(x))$
where $\mathscr{M}$ denotes the Hardy-Littlewood maximal operator and W_* the maximal operator associated to the classical heat semigroup.

Steps in the proof of this type of inequalities: Let $G(x) = K^{\mathcal{L}}_{\rm glob}(f)(x;\cdot,\cdot)$, $x \in \mathbb{R}^n$.

- $G(x) \in L^2(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B})$, $x \in \mathbb{R}^n$.
- $\|G(x)\|_{\gamma(H,\mathbb{B})} = \left(\mathbb{E}\left\|\sum_{j=1}^\infty \gamma_j T_{G(x)}(h_j)\right\|^2_{\mathbb{B}}\right)^{1/2}$

$$\leq C\int_{\mathbb{R}^n\setminus B(x,\rho(x))} \|f(z)\|_{\mathbb{B}} \|\chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t^{\mathcal{L}_\nu}(y,z)\|_H dz$$

$$\leq \cdots \leq C\mathscr{M}(\|f\|_{\mathbb{B}})(x), \quad x \in \mathbb{R}^n.$$

It is established that:

(1) $\|K^{\mathscr{L}_\nu}_{\rm glob}(f)(x;\cdot,\cdot)\|_{\gamma(H,\mathbb{B})} \leq C\mathscr{M}(\|f\|_{\mathbb{B}})(x)$.

(2) $\|D^{\mathscr{L}_\nu}(f)(x;\cdot,\cdot)\|_{\gamma(H,\mathbb{B})} \leq C(W_*(\|f\|_{\mathbb{B}})(x) + \mathscr{M}(\|f\|_{\mathbb{B}})(x))$
where $\mathscr{M}$ denotes the Hardy-Littlewood maximal operator and W_* the maximal operator associated to the classical heat semigroup.

Steps in the proof of this type of inequalities: Let $G(x) = K^{\mathscr{L}}_{\rm glob}(f)(x;\cdot,\cdot)$, $x \in \mathbb{R}^n$.

- $G(x) \in L^2(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B})$, $x \in \mathbb{R}^n$.
- $\|G(x)\|_{\gamma(H,\mathbb{B})} = \left(\mathbb{E}\left\|\sum_{j=1}^{\infty} \gamma_j T_{G(x)}(h_j)\right\|^2_{\mathbb{B}}\right)^{1/2}$

$$\leq C\int_{\mathbb{R}^n\setminus B(x,\rho(x))} \|f(z)\|_{\mathbb{B}} \|\chi_{\Gamma(x)} t^\beta \partial_t^\beta P_t^{\mathscr{L}_\nu}(y,z)\|_H dz$$

$$\leq \cdots \leq C\mathscr{M}(\|f\|_{\mathbb{B}})(x), \quad x \in \mathbb{R}^n.$$

It is established that:

(1) $\|K_{\text{glob}}^{\mathscr{L}_V}(f)(x;\cdot,\cdot)\|_{\gamma(H,\mathbb{B})} \leq C\mathscr{M}(\|f\|_{\mathbb{B}})(x)$.

(2) $\|D^{\mathscr{L}_V}(f)(x;\cdot,\cdot)\|_{\gamma(H,\mathbb{B})} \leq C(W_*(\|f\|_{\mathbb{B}})(x) + \mathscr{M}(\|f\|_{\mathbb{B}})(x))$
where $\mathscr{M}$ denotes the Hardy-Littlewood maximal operator and W_* the maximal operator associated to the classical heat semigroup.

Steps in the proof of this type of inequalities: Let $G(x) = K_{\text{glob}}^{\mathscr{L}}(f)(x;\cdot,\cdot)$, $x \in \mathbb{R}^n$.

- $G(x) \in L^2(\mathbb{R}^n \times (0,\infty), \frac{dydt}{t^{n+1}}; \mathbb{B})$, $x \in \mathbb{R}^n$.
- $\|G(x)\|_{\gamma(H,\mathbb{B})} = \left(\mathbb{E}\left\| \sum_{j=1}^{\infty} \gamma_j T_{G(x)}(h_j)\right\|_{\mathbb{B}}^2\right)^{1/2}$

$$\leq C \int_{\mathbb{R}^n \setminus B(x,\rho(x))} \|f(z)\|_{\mathbb{B}} \|\chi_{\Gamma(x)} t^{\beta}\partial_t^{\beta} P_t^{\mathscr{L}_V}(y,z)\|_H dz$$

$$\leq \cdots \leq C\mathscr{M}(\|f\|_{\mathbb{B}})(x), \quad x \in \mathbb{R}^n.$$

By using Theorem 1 for the operator $\mathbb{K}$, it is obtained that

- $\|\mathbb{K}_{\text{loc}}(f)\|_{L^p(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$.

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq \|t^\beta \partial_t^\beta P_t^{\mathcal{L}_\nu}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})}$$

- Polarization formula: Betancor, Fariña, R.-M., Testoni and Torrea, JMAA, 2012.
- Kalton and Weis's result; duality and density arguments.

Hermite setting: for every $n \in \mathbb{N}$ we obtain the necessary properties to follow the same argument as in the Schrödinger case.

By using Theorem 1 for the operator $\mathbb{K}$, it is obtained that

- $\|\mathbb{K}_{\rm loc}(f)\|_{L^p(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$.

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq \|t^\beta \partial_t^\beta P_t^{\mathscr{L}_V}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})}$$

- Polarization formula: Betancor, Fariña, R.-M., Testoni and Torrea, JMAA, 2012.
- Kalton and Weis's result; duality and density arguments.

Hermite setting: for every $n \in \mathbb{N}$ we obtain the necessary properties to follow the same argument as in the Schrödinger case.

By using Theorem 1 for the operator $\mathbb{K}$, it is obtained that

- $\|\mathbb{K}_{\mathrm{loc}}(f)\|_{L^p(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$.

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq \|t^\beta \partial_t^\beta P_t^{\mathscr{L}_V}(f)\|_{T_p^2(\mathbb{R}^n,\mathbb{B})}$$

- Polarization formula: Betancor, Fariña, R.-M., Testoni and Torrea, JMAA, 2012.
- Kalton and Weis's result; duality and density arguments.

Hermite setting: for every $n \in \mathbb{N}$ we obtain the necessary properties to follow the same argument as in the Schrödinger case.

∎

Consider now

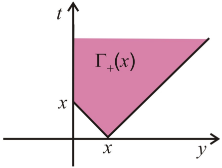
$$H_+ = L^2\left((0,\infty)^2, \frac{dydt}{t^2}\right)$$

Tent space $T_p^2((0,\infty),\mathbb{B})$

Let $1 \leq p < \infty$. The space $T_p^2((0,\infty),\mathbb{B})$ is defined as the completion of $C_c^\infty((0,\infty)^2) \otimes \mathbb{B}$ with respect to the norm

$$\|f\|_{T_p^2((0,\infty),\mathbb{B})} := \|J_+(f)\|_{L^p((0,\infty),\gamma(H_+,\mathbb{B}))},$$

where $[J_+(f)(x)](y,t) := \chi_{\Gamma_+(x)}(y,t)f(y,t), \quad x,y,t > 0.$



Bessel context

Bessel operator in $(0,\infty)$: $B_\lambda = -\dfrac{d^2}{dx^2} + \dfrac{\lambda(\lambda-1)}{x^2}$ $(\lambda > 0)$

The Hankel transform of $f \in L^1(0,\infty)$ is given by

$$h_\lambda(f)(x) = \int_0^\infty \sqrt{xy} J_{\lambda-1/2}(xy) f(y)\,dy, \quad x \in (0,\infty).$$

- $h_\lambda(B_\lambda f)(x) = x^2 h_\lambda(f)(x), \quad x > 0.$
- h_λ can be extended to $L^2(0,\infty)$ as an isometry in $L^2(0,\infty)$ and $h_\lambda^{-1} = h_\lambda$.

We define $\mathscr{B}_\lambda(f) = h_\lambda(x^2 h_\lambda(f))$, $f \in D(\mathscr{B}_\lambda)$, where

$$D(\mathscr{B}_\lambda) = \{f \in L^2(0,\infty) : y^2 h_\lambda(f) \in L^2(0,\infty)\}.$$

- Poisson semigroup associated to $\mathscr{B}_\lambda$:

$$P_t^{\mathscr{B}_\lambda}(f)(x) = \int_0^\infty P_t^{\mathscr{B}_\lambda}(x,y) f(y)\,dy, \quad f \in L^p(0,\infty),$$

$$P_t^{\mathscr{B}_\lambda}(x,y) = \frac{2\lambda(xy)^\lambda t}{\pi} \int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(t^2+(x-y)^2+2xy(1-\cos\theta))^{\lambda+1}}\,d\theta, \quad x,y,t > 0.$$

Bessel context

Bessel operator in $(0,\infty)$: $B_\lambda = -\dfrac{d^2}{dx^2} + \dfrac{\lambda(\lambda-1)}{x^2}$ $(\lambda > 0)$

The Hankel transform of $f \in L^1(0,\infty)$ is given by

$$h_\lambda(f)(x) = \int_0^\infty \sqrt{xy} J_{\lambda-1/2}(xy) f(y) dy, \quad x \in (0,\infty).$$

- $h_\lambda(B_\lambda f)(x) = x^2 h_\lambda(f)(x), \quad x > 0.$
- h_λ can be extended to $L^2(0,\infty)$ as an isometry in $L^2(0,\infty)$ and $h_\lambda^{-1} = h_\lambda$.

We define $\mathscr{B}_\lambda(f) = h_\lambda(x^2 h_\lambda(f))$, $f \in D(\mathscr{B}_\lambda)$, where

$$D(\mathscr{B}_\lambda) = \{f \in L^2(0,\infty) : y^2 h_\lambda(f) \in L^2(0,\infty)\}.$$

- Poisson semigroup associated to $\mathscr{B}_\lambda$:

$$P_t^{\mathscr{B}_\lambda}(f)(x) = \int_0^\infty P_t^{\mathscr{B}_\lambda}(x,y) f(y) dy, \quad f \in L^p(0,\infty),$$

$$P_t^{\mathscr{B}_\lambda}(x,y) = \frac{2\lambda (xy)^\lambda t}{\pi} \int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(t^2 + (x-y)^2 + 2xy(1-\cos\theta))^{\lambda+1}} d\theta, \quad x,y,t > 0.$$

Bessel context

Bessel operator in $(0,\infty)$: $B_\lambda = -\frac{d^2}{dx^2} + \frac{\lambda(\lambda-1)}{x^2}$ $(\lambda > 0)$

The Hankel transform of $f \in L^1(0,\infty)$ is given by

$$h_\lambda(f)(x) = \int_0^\infty \sqrt{xy} J_{\lambda-1/2}(xy) f(y)\, dy, \quad x \in (0,\infty).$$

- $h_\lambda(B_\lambda f)(x) = x^2 h_\lambda(f)(x), \quad x > 0.$
- h_λ can be extended to $L^2(0,\infty)$ as an isometry in $L^2(0,\infty)$ and $h_\lambda^{-1} = h_\lambda$.

We define $\mathscr{B}_\lambda(f) = h_\lambda(x^2 h_\lambda(f))$, $f \in D(\mathscr{B}_\lambda)$, where

$$D(\mathscr{B}_\lambda) = \{f \in L^2(0,\infty) : y^2 h_\lambda(f) \in L^2(0,\infty)\}.$$

- Poisson semigroup associated to $\mathscr{B}_\lambda$:

$$P_t^{\mathscr{B}_\lambda}(f)(x) = \int_0^\infty P_t^{\mathscr{B}_\lambda}(x,y) f(y)\, dy, \quad f \in L^p(0,\infty),$$

$$P_t^{\mathscr{B}_\lambda}(x,y) = \frac{2\lambda (xy)^\lambda t}{\pi} \int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(t^2 + (x-y)^2 + 2xy(1-\cos\theta))^{\lambda+1}}\, d\theta, \quad x,y,t > 0.$$

Bessel context

Bessel operator in $(0,\infty)$: $B_\lambda = -\frac{d^2}{dx^2} + \frac{\lambda(\lambda-1)}{x^2}$ $(\lambda > 0)$

The Hankel transform of $f \in L^1(0,\infty)$ is given by

$$h_\lambda(f)(x) = \int_0^\infty \sqrt{xy} J_{\lambda-1/2}(xy) f(y)\,dy, \quad x \in (0,\infty).$$

- $h_\lambda(B_\lambda f)(x) = x^2 h_\lambda(f)(x), \quad x > 0.$
- h_λ can be extended to $L^2(0,\infty)$ as an isometry in $L^2(0,\infty)$ and $h_\lambda^{-1} = h_\lambda$.

We define $\mathscr{B}_\lambda(f) = h_\lambda(x^2 h_\lambda(f))$, $f \in D(\mathscr{B}_\lambda)$, where

$$D(\mathscr{B}_\lambda) = \{f \in L^2(0,\infty) : y^2 h_\lambda(f) \in L^2(0,\infty)\}.$$

- Poisson semigroup associated to $\mathscr{B}_\lambda$:

$$P_t^{\mathscr{B}_\lambda}(f)(x) = \int_0^\infty P_t^{\mathscr{B}_\lambda}(x,y) f(y)\,dy, \quad f \in L^p(0,\infty),$$

$$P_t^{\mathscr{B}_\lambda}(x,y) = \frac{2\lambda(xy)^\lambda t}{\pi} \int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(t^2+(x-y)^2+2xy(1-\cos\theta))^{\lambda+1}}\,d\theta, \quad x,y,t > 0.$$

Laguerre context

Laguerre operator in $(0,\infty)$: $L_\alpha = -\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - 1/4}{x^2} \quad (\alpha > -1/2)$

- $L_\alpha \varphi_k^\alpha = 2(2k+\alpha+1)\varphi_k^\alpha, \qquad \varphi_k^\alpha(x) = c_{k,\alpha} e^{-x^2/2} x^{\alpha+1/2} \ell_k^\alpha(x^2)$

We define the operator $\mathscr{L}_\alpha(f) = 2\sum_{k=0}^{\infty}(2k+\alpha+1)\langle f, \varphi_k^\alpha\rangle \varphi_k^\alpha$, $f \in D(\mathscr{L}_\alpha)$,

where

$$D(\mathscr{L}_\alpha) = \{f \in L^2(0,\infty) : \sum_{k=0}^{\infty}(2k+\alpha+1)^2 |\langle f, \varphi_k^\alpha\rangle|^2 < \infty\}.$$

- Heat semigroup associated to $\mathscr{L}_\alpha$:

$$W_t^{\mathscr{L}_\alpha}(f)(x) = \int_0^\infty W_t^{\mathscr{L}_\alpha}(x,y) f(y)\,dy, \quad f \in L^p(0,\infty),$$

$$W_t^{\mathscr{L}_\alpha}(x,y) = \left(\frac{2e^{-2t}}{1-e^{-4t}}\right)^{1/2} \left(\frac{2xye^{-2t}}{1-e^{-4t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-2t}}{1-e^{-4t}}\right) e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-4t}}{1-e^{-4t}}}.$$

- Poisson semigroup associated to $\mathscr{L}_\alpha$: by subordination.

Laguerre context

Laguerre operator in $(0,\infty)$: $L_\alpha = -\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - 1/4}{x^2} \quad (\alpha > -1/2)$

- $L_\alpha \varphi_k^\alpha = 2(2k+\alpha+1)\varphi_k^\alpha, \qquad \varphi_k^\alpha(x) = c_{k,\alpha} e^{-x^2/2} x^{\alpha+1/2} \ell_k^\alpha(x^2)$

We define the operator $\mathscr{L}_\alpha(f) = 2\sum_{k=0}^{\infty}(2k+\alpha+1)\langle f, \varphi_k^\alpha\rangle \varphi_k^\alpha$, $f \in D(\mathscr{L}_\alpha)$,

where

$$D(\mathscr{L}_\alpha) = \{f \in L^2(0,\infty) : \sum_{k=0}^{\infty}(2k+\alpha+1)^2 |\langle f, \varphi_k^\alpha\rangle|^2 < \infty\}.$$

- Heat semigroup associated to $\mathscr{L}_\alpha$:

$$W_t^{\mathscr{L}_\alpha}(f)(x) = \int_0^\infty W_t^{\mathscr{L}_\alpha}(x,y) f(y)\,dy, \quad f \in L^p(0,\infty),$$

$$W_t^{\mathscr{L}_\alpha}(x,y) = \left(\frac{2e^{-2t}}{1-e^{-4t}}\right)^{1/2} \left(\frac{2xye^{-2t}}{1-e^{-4t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-2t}}{1-e^{-4t}}\right) e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-4t}}{1-e^{-4t}}}.$$

- Poisson semigroup associated to $\mathscr{L}_\alpha$: by subordination.

Laguerre context

Laguerre operator in $(0,\infty)$: $L_\alpha = -\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - 1/4}{x^2} \quad (\alpha > -1/2)$

- $L_\alpha \varphi_k^\alpha = 2(2k+\alpha+1)\varphi_k^\alpha, \qquad \varphi_k^\alpha(x) = c_{k,\alpha} e^{-x^2/2} x^{\alpha+1/2} \ell_k^\alpha(x^2)$

We define the operator $\mathscr{L}_\alpha(f) = 2\sum_{k=0}^{\infty}(2k+\alpha+1)\langle f, \varphi_k^\alpha\rangle \varphi_k^\alpha$, $f \in D(\mathscr{L}_\alpha)$,

where

$$D(\mathscr{L}_\alpha) = \{f \in L^2(0,\infty) : \sum_{k=0}^{\infty}(2k+\alpha+1)^2 |\langle f, \varphi_k^\alpha\rangle|^2 < \infty\}.$$

- Heat semigroup associated to $\mathscr{L}_\alpha$:

$$W_t^{\mathscr{L}_\alpha}(f)(x) = \int_0^\infty W_t^{\mathscr{L}_\alpha}(x,y) f(y)\,dy, \quad f \in L^p(0,\infty),$$

$$W_t^{\mathscr{L}_\alpha}(x,y) = \left(\frac{2e^{-2t}}{1-e^{-4t}}\right)^{1/2} \left(\frac{2xye^{-2t}}{1-e^{-4t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-2t}}{1-e^{-4t}}\right) e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-4t}}{1-e^{-4t}}}.$$

- Poisson semigroup associated to $\mathscr{L}_\alpha$: by subordination.

Laguerre context

Laguerre operator in $(0,\infty)$: $L_\alpha = -\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - 1/4}{x^2} \quad (\alpha > -1/2)$

- $L_\alpha \varphi_k^\alpha = 2(2k+\alpha+1)\varphi_k^\alpha, \qquad \varphi_k^\alpha(x) = c_{k,\alpha} e^{-x^2/2} x^{\alpha+1/2} \ell_k^\alpha(x^2)$

We define the operator $\mathscr{L}_\alpha(f) = 2\sum_{k=0}^{\infty}(2k+\alpha+1)\langle f, \varphi_k^\alpha\rangle \varphi_k^\alpha$, $f \in D(\mathscr{L}_\alpha)$,

where

$$D(\mathscr{L}_\alpha) = \{f \in L^2(0,\infty) : \sum_{k=0}^{\infty}(2k+\alpha+1)^2 |\langle f, \varphi_k^\alpha\rangle|^2 < \infty\}.$$

- Heat semigroup associated to $\mathscr{L}_\alpha$:

$$W_t^{\mathscr{L}_\alpha}(f)(x) = \int_0^\infty W_t^{\mathscr{L}_\alpha}(x,y) f(y)\,dy, \quad f \in L^p(0,\infty),$$

$$W_t^{\mathscr{L}_\alpha}(x,y) = \left(\frac{2e^{-2t}}{1-e^{-4t}}\right)^{1/2} \left(\frac{2xye^{-2t}}{1-e^{-4t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-2t}}{1-e^{-4t}}\right) e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-4t}}{1-e^{-4t}}}.$$

- Poisson semigroup associated to $\mathscr{L}_\alpha$: by subordination.

Theorem 3

Let $\mathbb{B}$ be a UMD space, $1 < p < \infty$ and $\alpha, \beta, \lambda > 0$. Denote by $\mathbb{L}$ any of the operators $\mathscr{B}_\lambda$ or $\mathscr{L}_\alpha$. Then,

$$\left\| t^\beta \partial_t^\beta P_t^{\mathbb{L}}(f) \right\|_{T_p^2((0,\infty),\mathbb{B})} \sim \|f\|_{L^p((0,\infty),\mathbb{B})}, \quad f \in L^p((0,\infty),\mathbb{B}).$$

Ideas in the proof:

- The Poisson semigroups are not Markovian.
- The property is established first for $f \in C_c^\infty(0,\infty) \otimes \mathbb{B}$.
- For $f \in C_c^\infty(0,\infty) \otimes \mathbb{B}$ consider f_o the odd extension of f to $\mathbb{R}$ and $f_\chi = f_o \chi_{(0,\infty)}$. We write

$$t^\beta \partial_t^\beta P_t^{\mathscr{B}_\lambda}(f) = t^\beta \partial_t^\beta [P_t^{\mathscr{B}_\lambda}(f) - P_t(f_o)] + t^\beta \partial_t^\beta P_t(f_o)$$

$$t^\beta \partial_t^\beta P_t^{\mathscr{L}_\alpha}(f) = t^\beta \partial_t^\beta [P_t^{\mathscr{L}_\alpha}(f) - P_t^{\mathscr{H}}(f_\chi)] + t^\beta \partial_t^\beta P_t^{\mathscr{H}}(f_\chi).$$

- Polarization formula: Bessel (Hankel transform); Laguerre (spectral decomposition of $P_t^{\mathscr{L}_\alpha}(f)$).

  Kalton and Weis's result; duality and density arguments. ∎

Theorem 3

Let $\mathbb{B}$ be a UMD space, $1 < p < \infty$ and $\alpha, \beta, \lambda > 0$. Denote by $\mathbb{L}$ any of the operators $\mathscr{B}_\lambda$ or $\mathscr{L}_\alpha$. Then,

$$\left\| t^\beta \partial_t^\beta P_t^{\mathbb{L}}(f) \right\|_{T_p^2((0,\infty),\mathbb{B})} \sim \left\| f \right\|_{L^p((0,\infty),\mathbb{B})}, \quad f \in L^p((0,\infty),\mathbb{B}).$$

Ideas in the proof:

- The Poisson semigroups are not Markovian.
- The property is established first for $f \in C_c^\infty(0,\infty) \otimes \mathbb{B}$.
- For $f \in C_c^\infty(0,\infty) \otimes \mathbb{B}$ consider f_o the odd extension of f to $\mathbb{R}$ and $f_\chi = f_o \chi_{(0,\infty)}$. We write

$$t^\beta \partial_t^\beta P_t^{\mathscr{B}_\lambda}(f) = t^\beta \partial_t^\beta [P_t^{\mathscr{B}_\lambda}(f) - P_t(f_o)] + t^\beta \partial_t^\beta P_t(f_o)$$
$$t^\beta \partial_t^\beta P_t^{\mathscr{L}_\alpha}(f) = t^\beta \partial_t^\beta [P_t^{\mathscr{L}_\alpha}(f) - P_t^{\mathscr{H}}(f_\chi)] + t^\beta \partial_t^\beta P_t^{\mathscr{H}}(f_\chi).$$

- Polarization formula: Bessel (Hankel transform); Laguerre (spectral decomposition of $P_t^{\mathscr{L}_\alpha}(f)$).

  Kalton and Weis's result; duality and density arguments. ■

Merci beaucoup!!!