# On subordination of holomorphic semigroups

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# Subordination

**Definition** A family of positive Borel measures $(\mu_t)_{t\geq 0}$ on $\mathbb{R}_+$ is called a vaguely continuous convolution semigroup of subprobability measures ("subordination semigroup") if for all $s, t \geq 0$ :

$$\mu_t(\mathbb{R}_+) \leq 1, \qquad \mu_{t+s} = \mu_t * \mu_s, \qquad \text{and} \qquad \mathrm{vague} - \lim_{t\to 0+} \mu_t = \delta_0.$$

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**Theorem**(Bochner subordination) A function $\varphi$ on $\mathbb{R}_+$ is called Bernstein if and only if there exists a subordination semigroup of measures $(\mu_t)_{t\geq 0}$ on $\mathbb{R}_+$ such that for all $t \geq 0$ :

$$e^{-t\varphi(z)} = \int_0^\infty e^{-sz}\, d\mu_t(s), \qquad z \in \mathbb{C}_+.$$

# Semigroup framework

Given: $(e^{-tA})_{t\geq 0}$ is a $C_0$-semigroup on a Banach space $X$ with generator $-A$,
$(\mu_t)_{t\geq 0}$ is a subordination semigroup of measures.
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The essential feature: $(e^{-tA^\alpha})_{t\geq 0}$ are holomorphic.

# Examples

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**3. More generally**: $\mu$ is a probability measure, then

$$\mu_t = e^{-t} \sum_{n=0}^\infty \frac{t^n}{n!} \mu^{*n}, \quad t \geq 0.$$

is a a subordination semigroup.

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# Bernstein functions, $\mathcal{BF}$

A function $\varphi : (0, \infty) \mapsto (0, \infty)$ is *completely monotone* if there exists a positive measure $\mu$ such that

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A function $\psi$ is Bernstein $\Leftrightarrow$ $a, b \geq 0$ and a positive Radon measure $\mu$ on $(0,\infty)$ :

$$\int_{0+}^\infty \frac{s}{1+s}\,\mu(ds) < \infty$$

such that the Lévy-Hintchine representation holds:

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Every Bernstein function extends holomorphically to $\mathbb{C}_+$ and continuously to $\overline{\mathbb{C}}_+$

# More on Bernstein

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**Proposition** Let $F \in \mathcal{BF}$. Then $F$ preserves angular sectors, i.e.

$$F(\overline{\Sigma}_{\omega}) \subset \overline{\Sigma}_{\omega}, \qquad \omega \in (0, \pi/2),$$

where $\Sigma_{\omega} := \{z : |\arg z| < \omega\}$.

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**Definition** A Bernstein function $\psi$ is *complete* if the measure $\mu$ in its Lévy-Hintchine representation has a completely monotone density wrt to Lebesgue measure.

**Theorem** Let $\psi$ be a non-negative function on $(0, \infty)$. TFAE.

(i) $\psi \in \mathcal{CBF}$,

(ii) $\exists$ a Bernstein $\varphi$ :

$$\psi(\lambda) = \lambda^2 \widehat{\varphi}(\lambda), \quad \lambda > 0.$$

(iii) $\psi(\{\text{upper halfplane}\}) \subset \{\text{upper halfplane}\}$ and $\exists \psi(0+) = \lim_{\lambda \to 0+} \psi(\lambda)$

(iv) $\exists a, b \geq 0$ and a Borel $\sigma$ :

$$\psi(\lambda) = a + b\lambda + \int_{0+}^{\infty} \frac{\lambda\, \sigma(ds)}{\lambda + s}, \qquad \int_{0+}^{\infty} \frac{\sigma(ds)}{1+s} < \infty.$$

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$$\varphi(\lambda) = \lambda^{-1}\widehat{\psi}(\lambda^{-1}), \quad \lambda > 0.$$

**Theorem** Let $\psi \in \mathcal{BF}$ and let $\varphi \in \mathcal{CBF}$ be associated with $\psi$. Let $\omega \in (\pi/2, \pi)$ and $z \in \Sigma_\omega$ be fixed. Define

$$r(\lambda; z) := \frac{1}{z + \psi(\lambda)} - \frac{1}{z + \varphi(\lambda)}, \qquad \lambda \in \Sigma_{\pi-\omega}.$$

Then $r(\cdot; z)$ is holomorphic in $\Sigma_{\pi-\omega}$ and for every $\beta \in (0, \pi - \omega)$ :

$$\int_{\partial\Sigma_\beta} |r(\lambda; z)| \frac{|d\lambda|}{|\lambda|} \le \frac{8}{\cos^2\beta \cos^2((\omega+\beta)/2)\,|z|}, \qquad z \in \Sigma_\omega.$$

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All of the calculi are compatible and eventually allow one to plug in operator into relevant formulas.

# An illustration

**Theorem** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t\geq 0}$ on $X$,
and let $\psi$ be a Bernstein function with the Lévy-Hintchine repr. $(a, b, \mu)$.

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(i) Then $\psi(A)|_{\mathrm{dom}(A)}$ is given by

$$\psi(A)x = ax + bAx + \int_{0+}^{\infty}(1 - e^{-sA})x\,\mu(ds), \qquad x \in \mathrm{dom}(A),$$

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(ii) Moreover, $-\psi(A)$ is the generator of a bounded $C_0$-semigroup $(e^{-t\psi(A)})_{t\geq 0}$ on $X$ given by

$$e^{-t\psi(A)} := \int_0^{\infty} e^{-sA}\,\mu_t(ds), \qquad t \geq 0,$$

where $(\mu_t)_{t\geq 0}$ is a subordinator corresponding to $\psi$.

# Answer for $\mathcal{CBF}$

**Theorem** Let $A \in \mathrm{Sect}(\alpha)$, $\alpha \in (0, \pi)$. If $\psi \in \mathcal{CBF}$, then $\psi(A) \in \mathrm{Sect}(\alpha)$ too.

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Choose $q \in (1, \pi/\alpha)$ so that $1/q \in (0, 1)$,
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The crux matter: $\psi \in \mathcal{CBF}(\mathcal{BF}?) \Longrightarrow z^q \circ \psi \circ z^{1/q} \in \mathcal{CBF}$

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**Proposition** Let $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$. Then

$$(z + \psi(A))^{-1} = (z + \varphi(A))^{-1} + r(A; z), \quad z \in \Sigma_\omega,$$

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where for every $\omega \in (\pi/2, \pi/2 + \theta)$,
$r(A, \cdot)$ is holomorphic in $\Sigma_\omega$ and for every $\beta \in (\pi/2 - \theta, \pi - \omega)$,

$$\|r(A; z)\| \le \frac{4M(A, \pi - \beta)}{\pi \cos^2 \beta \, \cos^2((\gamma + \beta)/2)\, |z|}, \quad z \in \Sigma_\omega.$$

# Main Results, I

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**Theorem (“yes” to KR question)** Let $-A$ be the generator of a bounded $C_0$-semigroup on $X$ such that $-A \in \mathcal{H}(\theta)$ for some $\theta \in (0, \pi/2]$. Then for every $\psi \in \mathcal{BF}$ one has $-\psi(A) \in \mathcal{H}(\theta)$.

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**Theorem** Suppose that $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$ and ran($A$) is dense. If $h = \psi + f$, where $\psi \in \mathcal{BF}$ and $1/f \in \mathcal{BF}$, then $-h(A) \in \mathcal{BH}(\theta)$.

# “Improving” properties of subordination: Carasso-Kato, Fujita, Mirotin, ...

Given: For a fixed Bernstein function $\psi$ the mapping

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A typical result:

**Theorem [Carasso-Kato]** Each subordination semigroup $[0, \infty) \ni t \mapsto \mu_t$ such that $\mu_t$ is continuously differentiable in $M_b([0, \infty))$ for $t > 0$, with

$$\|\mu_t{}'\|_{M_b(\mathbb{R}_+)} = \mathrm{O}(t^{-1}) \quad \text{as} \quad t \to 0+,$$

gives rise to improving $\psi$ via Bochner's subordination.

# Main results (geometric conditions !), II

**Theorem** Let $\psi$ be a *complete* Bernstein function and let $\gamma \in (0, \pi/2)$ be fixed. TFAE.

(i) $\psi(\overline{\mathbb{C}}_+) \subset \overline{\Sigma}_\gamma$.

(ii) For each (complex) Banach space $X$ and each generator $-A$ of a bounded $C_0$-semigroup on $X$, one has $-\psi(A) \in \mathcal{BH}(\pi/2 - \gamma)$.

# Further geometric conditions: a flavour

**Theorem** Let $\psi$ be a Bernstein function. Suppose there exist $\theta \in (\pi/2, \pi)$ and $r > 0$ such that $\psi$ admits a continuous extension to $\overline{\Sigma}_\theta$ which is holomorphic in $\Sigma_\theta$, and

$$0 \le \arg(\psi(\lambda)) \le \pi/2 \qquad \text{if} \qquad 0 \le \arg(\lambda) \le \theta \quad \text{and} \quad |\lambda| \ge r.$$

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If $-A$ is the generator of a bounded $C_0$-semigroup on $X$, then $A \in \mathcal{H}(\frac{\pi}{2}(1 - \frac{\pi}{2\theta}))$.

# Summary

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for any Bernstein function $\psi$ the operator $-\psi(A)$ generates a holomorphic $C_0$-semigroup $(e^{-t\psi(A)})_{t\geq 0}$ on a Banach space, whenever $-A$ does.

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4. Still there's a number of problems to solve here ...