

Karlsruher Institut für Technologie



R-boundedness versus γ -boundedness

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joint work with Stanisław Kwapien and Mark Veraar

Workshop on

Functional calculus and Harmonic analysis of semigroups

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Square Functions and Random Sums

$$X = Y = L^p(\Omega, \mu) \quad , \quad T_j \in B(X) \quad , \quad x_j \in X$$

$$(1) \quad \left\| \left(\sum_j |T_j x_j|^2 \right)^{1/2} \right\|_{L^p} \leq C \left\| \left(\sum_j |x_j|^2 \right)^{1/2} \right\|_{L^p}$$

Marcinkiewicz-Zygmund 1939 , $1 \leq p < \infty$

$$\left\| \left(\sum_j |x_j|^2 \right)^{1/2} \right\|_{L^p} \cong \mathbb{E} \left\| \sum_j \gamma_j x_j \right\|_{L^p} \cong \mathbb{E} \left\| \sum_j r_j x_j \right\|_{L^p}$$

$$(2) \quad \mathbb{E} \left\| \sum_j \gamma_j T_j x_j \right\|_Y \leq C \mathbb{E} \left\| \sum_j \gamma_j x_j \right\|_X$$

(γ_j) independent $N(0,1)$ RV

$$(3) \quad \mathbb{E} \left\| \sum_j r_j T_j x_j \right\|_Y \leq C \mathbb{E} \left\| \sum_j r_j x_j \right\|_X$$

(r_j) Rademacher or Bernoulli RV

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R-, γ -, and R_2 - boundedness

Definition: X, Y Banach spaces, $\tau \subset B(X, Y)$

- τ is called **γ -bounded** (resp. **R-bounded**) if there is a constant C such that (2) (resp. (3)) holds for all $T_j \in \tau, x_j \in X$.
- If X and Y are B-lattices, then (1) defines **R_2 -boundedness**.
- The best constant in (1), (2), (3) are denoted, resp., by $R^L(\tau), R^\gamma(\tau), R_2(\tau)$.

Remark: a) If X has finite cotype, then

$$\begin{aligned} \mathbb{E} \left\| \sum_j \gamma_j x_j \right\|_X &\approx \mathbb{E} \left\| \sum_j r_j x_j \right\|_X \\ &\approx \left\| \left(\sum_j |x_j|^2 \right)^{1/2} \right\|_X \quad \text{if } X \text{ is a Banach function space.} \end{aligned}$$

b) R -boundedness implies γ -boundedness, always.

Today's result: All other implications fail.

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Calderon Zygmund Theory

- Given $t \in \mathbb{R} \setminus \{0\} \rightarrow M(t) \in B(X)$.
- Define Fourier multiplier operators T_M by

$$(T_M f)^\wedge(t) = M(t) \hat{f}(t) \quad , \quad f \in \mathcal{S}(\mathbb{R}, X) .$$

Theorem: If X is a UMD space and

$$\{M(t), tM'(t) : t \in \mathbb{R} \setminus \{0\}\}$$

is \mathbb{R} -bounded, then $T_M: L^p(\mathbb{R}, X) \rightarrow L^p(\mathbb{R}, X) \quad , \quad 1 < p < \infty .$

- Marcinkiewicz, Mihlin, Hörmander multiplier theorems.
 \mathbb{R}^n : Strkalj-W., Girardi-W., Haller-Heck-Noll, Hytonen
 \mathbb{T} : Arendt-Bu
- Pseudodifferential op.: Strkalj-Portal, Portal-Hytonen, Denk-Krainer
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H^∞ -Functional Calculus

Let A have a $H^\infty(\Sigma_\sigma)$ -functional calculus on X , i.e. for all $x \in X$

$$Ax = \sum_{k \in \mathbb{Z}} \varphi(2^k A) Ax \quad , \quad \varphi \in H_0^\infty(\Sigma_\sigma) \quad , \quad \sum_k \varphi(2^k \lambda) \equiv 1$$

$$\|x\| \cong \mathbb{E} \left\| \sum_{k \in \mathbb{Z}} r_k \varphi(2^k A) x \right\|_X .$$

The role of R-boundedness (cf. Kalton-W.)

- $\{e^{-\gamma|t|} A^{it} : t \in \mathbb{R}\}$ γ -bnd $\xrightarrow{\sigma > \gamma}$ A has $H^\infty(\Sigma_\sigma)$ calculus
 $\Rightarrow \{e^{-\sigma t} A^{it} : t \in \mathbb{R}\}$ R-bnd (if X has property (α))

- $\omega_{H^\infty}(A) = \omega_R(A)$ (if X has property (Δ))

- Operator-valued functional calculus

$$F(A) = \frac{1}{2\pi i} \int_{\partial \Sigma_\gamma} F(\lambda) R(\lambda, A) d\lambda \text{ for } \lambda \in \Sigma_\sigma \rightarrow F(\lambda) \in B(X)$$

resolvent commuting with A and $\{F(\lambda) : \lambda \in \Sigma_\sigma\}$ is R-bounded.

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Spectral Multiplier Theorem

Let A be a 0-sectorial operator on X , $\sigma(A) \subset \mathbb{R}_+$, with $H^\infty(\Sigma_\sigma)$, $\sigma > 0$.

Hörmander condition $\mathcal{H}^{\alpha,p}$, $p \in [1, \infty]$, $\alpha > \frac{1}{p}$, for $f: \mathbb{R}_+ \rightarrow \mathbb{C}$,

$$\sup_{R>0} \int_R^{2R} \left| t^n \frac{d^n}{dt^n} f(t) \right|^p \frac{dt}{t} < \infty, \quad n \leq \alpha.$$

$\mathcal{H}^{\alpha,p}$ functional calculus: $f \in \mathcal{H}^{\alpha,p} \Rightarrow f(A) \in B(X)$.

Conditions for $\mathcal{H}^{\alpha,p}$ in terms of γ -boundedness (Kriegler-W), X prop (α)

- $\left\{ (1 + |t|)^{-\beta} A^{it} : t \in \mathbb{R} \right\}$ γ -bounded $\Rightarrow \mathcal{H}^{\alpha,2}$ for $\alpha > \beta + \frac{1}{2}$
- $R^\gamma \left(\left\{ \exp(-e^{i\theta} tA) : t > 0 \right\} \right) \lesssim \left(\frac{\pi}{2} - \theta \right)^{-\beta} \Rightarrow \mathcal{H}^{\alpha,2}$, $\alpha > \beta + \frac{1}{2}$
- $\left\{ R_u^{\alpha-1}(A) : u > 0 \right\}$ R-bounded $\Rightarrow A$ has $\mathcal{H}^{\alpha,1}$ calculus.
- " $\Leftarrow A$ has $\mathcal{H}^{\alpha+\varepsilon,1}$ calculus.

Bochner-Riesz means: $R_u^{\alpha-1}(\lambda) = \left(1 - \frac{\lambda}{u} \right)_+^{\alpha-1}$, $\alpha > 1$

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How to Check R-boundedness

$$t \in G \rightarrow \sigma(t) \in B(X), \quad \text{e.g.} = L^p(\Omega, \mu)$$

- (generalized) Gaussian-, Poisson bounds on $T(t)$
 Hieber-Prüß, Coulhon-Lamberton
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- Maximal inequalities, e.g. $\|(\sum_j |Mf_j|^2)^{1/2}\|_{L^p} \lesssim \|(\sum_j |f_j|^2)^{1/2}\|_{L^p}$
- Contractivity and Positivity, e.g. Diffusionsemigroups
- Averaging (Hytonen, Veraar)
 $\int_I \|T(t)\|^r dt < \infty$ on X with $1/r > 1/p_X - 1/q_Y$
 $\Rightarrow \left\{ \int_I f(t) T(t) dt : \|f\|_{L^{r'}} \leq 1 \right\}$ R-bnd
- Smoothness (Girardi-W., Hytonen-Veraar)
 $T(\cdot) \in B_{r,1}^{d/r}(\mathbb{R}^d, B(X))$ implies R-boundedness for
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Bootstrapping with the Functional Calculus

X has property (α)

- Let A have a $H^\infty(\Sigma_\sigma)$ calculus on X .

Then $\{f(A) : \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\}$ is R-bounded.

- Let A have a $\mathcal{H}^{\alpha,2}$ calculus on X .

Then $\{f(A) : \|f\|_{\mathcal{H}^{\beta,2}} \leq 1\}$ is R-bounded for $\beta > \alpha + \frac{1}{2}$.

Kalton, Girardi, Kriegler, W.

Kalton's Factorization Theorem

X Banach space

$\tau \in B(X)$ γ -bounded, absolutely convex, closed in strong operator top.

Then there exists

- a Hilbert space H and a closed subalgebra $B \subset B(H)$
- $u: \tilde{\tau} = (\text{span } \tau, \|\cdot\|_{\tau}) \rightarrow B$ linear, bounded
- $v: B \rightarrow B(X)$ bounded algebra homomorphism

such that

$$(v \circ u)(T) = T$$

for all $T \in \tilde{\tau}$.

R-boundedness versus γ -boundedness

X, Y be Banach spaces with $Y \neq \{0\}$

Theorem (Kwapień, Veraar, W.)

Each γ -bounded family $\tau \subset B(X, Y)$ is R-bounded if and only if X has finite cotype.

Remarks:

- R-boundedness \Rightarrow γ -boundedness, always.
 (γ_n) and $(r_n \gamma_n)$ have the same distribution
- X finite cotype: R-bnd \iff γ -bnd

Proof: " \Rightarrow " Assume X not finite cotype. We will construct families $\tau_N \subset B(X, Y)$ with $R^\gamma(\tau_N) / R(\tau_N) \xrightarrow{N \rightarrow \infty} 0$.

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Proof of the Theorem Continued

$$\begin{array}{ccc}
 X & & Y \\
 P_N \downarrow & & \uparrow i \\
 X_N & \xrightarrow{J_n} & \ell_N^\infty \xrightarrow{T} \mathbb{R}
 \end{array}
 \quad \|P_N\|, \|J_N\|, \|J_N^{-1}\| \leq C$$

Construct $\tau_N \subset B(\ell_N^\infty, \mathbb{R})$ with

(a) $R(\tau_N) \sim N^{1/2}$

(b) $R^\gamma(\tau_N) \sim \left(\frac{N}{\log N}\right)^{1/2}$.

For a fixed N choose $\tau_N = \{T_n((x_1, \dots, x_N)) = x_n : n = 1, \dots, N\}$.

Part (a). Let e_n be the unit vectors of ℓ_N^∞ .

$$N^{1/2} = \left(\sum_{n=1}^N |T_n(e_n)|^2 \right)^{1/2} = \mathbb{E} \left| \sum_{n=1}^N r_n T_n e_n \right| \leq R(\tau_N) \underbrace{\mathbb{E} \left\| \sum_{n=1}^N r_n e_n \right\|_{\ell^\infty}}_{=1}$$

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Part (b). For $S_1, \dots, S_J \in \{T_1, \dots, T_N\}$ and $x_1, \dots, x_J \in \ell_N^\infty$

$$\left(\mathbb{E} \left| \sum_{j=1}^J \gamma_j S_j(x_j) \right|^2 \right)^{1/2} \leq 4 \left(\frac{N}{\log N} \right)^{1/2} \left(\mathbb{E} \left\| \sum_{j=1}^J \gamma_j x_j \right\|_2^2 \right)^{1/2}. \quad (1)$$

$$A_n := \{j: S_j = T_n\}, \quad n = 1, \dots, N.$$

Put $a_n := \left(\sum_{j \in A_n} |T_n(x_j)|^2 \right)^{1/2}$ and $\Gamma_n := \sum_{j \in A_n} \gamma_j x_j \in \ell_N^\infty$.

Then (1) takes the form

$$\left(\sum_{n=1}^N a_n^2 \right)^{1/2} \leq 4 \left(\frac{N}{\log N} \right)^{1/2} \left(\mathbb{E} \left\| \sum_{n=1}^N \Gamma_n \right\|_\infty^2 \right)^{1/2}. \quad (2)$$

For (2) we need for $\alpha_1, \dots, \alpha_n \in \mathbb{R}$

$$\left(\frac{\log N}{N} \sum_{n=1}^N \alpha_n^2 \right)^{1/2} \leq 4 \mathbb{E} \sup_{n \leq N} |\alpha_n \gamma_n| \quad (3)$$

Proof Continued

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$$\left(\sum_{n=1}^N a_n^2 \right)^{1/2} \leq 4 \left(\frac{N}{\log N} \right)^{1/2} \left(\mathbb{E} \left\| \sum_{n=1}^N \Gamma_n \right\|_\infty^2 \right)^{1/2}. \quad (2)$$

For (2) we need for $\alpha_1, \dots, \alpha_n \in \mathbb{R}$

$$\left(\frac{\log N}{N} \sum_{n=1}^N \alpha_n^2 \right)^{1/2} \leq 4 \mathbb{E} \sup_{n \leq N} |\alpha_n \gamma_n| \quad (3)$$

Proof Continued

Part (b). For $S_1, \dots, S_J \in \{T_1, \dots, T_N\}$ and $x_1, \dots, x_J \in \ell_N^\infty$

$$\left(\mathbb{E} \left| \sum_{j=1}^J \gamma_j S_j(x_j) \right|^2 \right)^{1/2} \leq 4 \left(\frac{N}{\log N} \right)^{1/2} \left(\mathbb{E} \left\| \sum_{j=1}^J \gamma_j x_j \right\|_2^2 \right)^{1/2}. \quad (1)$$

$$A_n := \{j : S_j = T_n\}, \quad n = 1, \dots, N.$$

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Proof (still) Continued

$$\sum_{n=1}^N |a_n|^2 \leq \left(\frac{16N}{\log N} \right) \mathbb{E} \sup |\gamma_n a_n|^2 = \mathbb{E} \sup |\Gamma_n(n)| \quad (4)$$

$(\Gamma_n(n))_{n=1}^N$ and $(\gamma_n a_n)_{n=1}^N$ are Gaussian with the same distribution since

$$\mathbb{E} |\Gamma_n(n)|^2 = \mathbb{E} \left| \sum_{j \in A_n} \gamma_j T_n(x_j) \right|^2 = a_n^2, \quad n = 1, \dots, N$$

Pointwise on Ω_γ we have

$$\sup_{1 \leq n \leq N} |\Gamma_n(n)| = \sup_{1 \leq n \leq N} \left\| \mathbb{E}_r \left[\sum_{m=1}^N r_m r_n(\Gamma_n)(m) \right] \right\|^2 = \left\| \mathbb{E}_r \left[I_n \left(\sum_{m=1}^N r_m \Gamma_m \right) \right] \right\|_\infty^2$$

where $I_n(\alpha_k) = (r_n(\cdot) \alpha_k)$ is an isometry on ℓ_N^∞

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Continue (4) with this estimate. Since $\Gamma_1, \dots, \Gamma_N$ indep.

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R- and γ - boundedness versus R_2 -boundedness

Theorem. Let X , Y be Banach lattices. TFAE.

- Every R_2 -bounded family $\tau \subset B(X, Y)$ is R-bounded.
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Remarks on R_2 -boundedness

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Duality

Notation: For $\tau \subset B(X, Y)$ put $\tau^* = \{T^* : T \in \tau\} \subset B(Y^*, X^*)$.

Theorem. For Banach spaces X, Y and $\tau \in B(X, Y)$, TFAE.

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