Karlsruher Institut für Technologie

R-boundedness versus γ-boundedness

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joint work with Stanisław Kwapien and Mark Veraar

Workshop on

Functional calculus and Harmonic analysis of semigroups

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Square Functions and Random Sums

$$X \;=\; Y \;=\; L^p(\Omega,\mu) \quad , \quad T_j \in B(X) \quad , \quad x_j \in X$$

(1) $$\left\| \left(\sum_j |T_j x_j|^2 \right)^{1/2} \right\|_{L^p} \;\leq\; C \; \left\| \left(\sum_j |x_j|^2 \right)^{1/2} \right\|_{L^p}$$

Marcinkiewicz-Zygmund 1939 , $1 \leq p < \infty$

$$\left\| \left(\sum_j |x_j|^2 \right)^{1/2} \right\|_{L^p} \;\cong\; \mathbb{E} \left\| \sum_j \gamma_j x_j \right\|_{L^p} \;\cong\; \mathbb{E} \left\| \sum_j r_j x_j \right\|_{L^p}$$

(2) $$\mathbb{E} \left\| \sum_j \gamma_j T_j x_j \right\|_Y \;\leq\; C\, \mathbb{E} \left\| \sum_j \gamma_j x_j \right\|_X$$

(γ_j) independent N(0,1) RV

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R-, γ-, and R$_2$- boundedness

Definition: X, Y Banch spaces , $\tau \subset B(X, Y)$

- τ is called γ-**bounded** (resp. **R-bounded**) if there is a constant C such that (2) (resp. (3)) holds for all $T_j \in \tau$, $x_j \in X$.
- If X and Y are B-lattices, then (1) defines **R$_2$-boundedness**.
- The best constant in (1), (2), (3) are denoted, resp., by $R^L(\tau)$, $R^\gamma(\tau)$, $R_2(\tau)$.

Remark: a) If X has finite cotype, then

$$\mathbb{E}\left\|\sum_j \gamma_j x_j\right\|_X \approx \mathbb{E}\left\|\sum_j r_j x_j\right\|_X$$

$$\approx \left\|\left(\sum_j |x_j|^2\right)^{1/2}\right\|_X \quad \text{if } X \text{ is a Banach function space .}$$

b) R-boundedness implies γ-boundedness, always.

Today's result: All other implications fail.

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Calderon Zygmund Theory

- Given $t \in \mathbb{R} \setminus \{0\} \longrightarrow M(t) \in B(X)$.
- Define Fourier multiplier operators T_M by

$$(T_M f)^\frown(t) = M(t)\widehat{f}(t) \quad , \quad f \in \mathcal{S}(\mathbb{R}, X) \ .$$

Theorem: If X is a UMD space and

$$\{M(t), tM'(t) : t \in \mathbb{R} \setminus \{0\}\}$$

is R-bounded, then $T_M : L^P(\mathbb{R}, X) \to L^P(\mathbb{R}, X) \quad , \quad 1 < p < \infty$.

- Marcinkiewicz, Mihlin, Hörmander multiplier theorems.

  $\mathbb{R}^n$: Strkalj-W., Girardi-W., Haller-Heck-Noll, Hytonen

  $\mathbb{T}$: Arendt-Bu
- Pseudodifferential op.: Strkalj-Portal, Portal-Hytonen, Denk-Krainer
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H^∞-Functional Calculus

Let A have a $H^\infty(\Sigma_\sigma)$-functional calculus on X, i.e. for all $x \in X$

$Ax = \sum_{k\in\mathbb{Z}} \varphi\left(2^k A\right) Ax \quad , \quad \varphi \in H_0^\infty(\Sigma_\sigma) \quad , \quad \sum_k \varphi\left(2^k\lambda\right) \equiv 1$

$$\|x\| \cong \mathbb{E}\left\|\sum_{k\in\mathbb{Z}} r_k \varphi\left(2^k A\right) x\right\|_X .$$

The role of R-boundedness (cf. Kalton-W.)

- $\left\{e^{-\gamma|t|}A^{it} : t \in \mathbb{R}\right\}$ γ-bnd $\overset{\sigma>\gamma}{\Longrightarrow}$ A has $H^\infty(\Sigma_\sigma)$ calculus
  $\Rightarrow \left\{e^{-\sigma t}A^{it} : t \in \mathbb{R}\right\}$ R-bnd (if X has property (α))
- $\omega_{H^\infty}(A) = \omega_R(A)$ (if X has property (Δ))
- Operator-valued functional calculus
  $F(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma} F(\lambda) R(\lambda, A)\, d\lambda$ for $\lambda \in \Sigma_\sigma \to F(\lambda) \in B(X)$
  resolvent commuting with A and $\{F(\lambda) : \lambda \in \Sigma_\sigma\}$ is R-bounded.

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Spectral Multiplier Theorem

Let A be a 0-sectorial operator on X, $\sigma(A) \subset \mathbb{R}_+$, with $H^\infty(\Sigma_\sigma)$, $\sigma > 0$.

Hörmander condition $\mathcal{H}^{\alpha,p}$, $p \in [1,\infty]$, $\alpha > \frac{1}{p}$, for $f : \mathbb{R}_+ \to \mathbb{C}$,

$$\sup_{R>0} \int_R^{2R} \left| t^n \frac{d^n}{dt^n} f(t) \right|^p \frac{dt}{t} < \infty \quad , \quad n \leq \alpha .$$

$\mathcal{H}^{\alpha,p}$ functional calculus: $f \in \mathcal{H}^{\alpha,p} \Rightarrow f(A) \in B(X)$.

Conditions for $\mathcal{H}^{\alpha,p}$ in terms of γ-boundedness (Kriegler-W), X prop (α)

- $\left\{(1+|t|)^{-\beta} A^{it} : t \in \mathbb{R}\right\}$ γ-bounded $\Rightarrow$ $\mathcal{H}^{\alpha,2}$ for $\alpha > \beta + \frac{1}{2}$
- $R^\gamma\left(\left\{\exp\left(-e^{i\theta} tA\right) : t > 0\right\}\right) \lesssim \left(\frac{\pi}{2} - \theta\right)^{-\beta} \Rightarrow \mathcal{H}^{\alpha,2}$, $\alpha > \beta + \frac{1}{2}$
- $\left\{R_u^{\alpha-1}(A) : u > 0\right\}$ R-bounded $\Rightarrow$ A has $\mathcal{H}^{\alpha,1}$ calculus.
- " $\Leftarrow$ A has $\mathcal{H}^{\alpha+\varepsilon,1}$ calculus.

Bochner-Riesz means: $R_u^{\alpha-1}(\lambda) = \left(1 - \frac{\lambda}{u}\right)_+^{\alpha-1} \quad , \quad \alpha > 1$

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How to Check R-boundedness

$$t \in G \;\rightarrow\; \sigma(t) \in B(X), \quad \text{e.g. } = L^p(\Omega, \mu)$$

- (generalized) Gaussian-, Poisson bounds on $T(t)$
  Hieber-Prüß, Coulhon-Lamberton
  Blunk, Kunstmann, Uhl, Auscher-Martell, Bernicot
- Maximal inequalities, e.g. $\|(\sum_j |Mf_j|^2)^{1/2}\|_{L^p} \lesssim \|(\sum_j |f_j|^2)^{1/2}\|_{L^p}$
- Contractivity and Positivity, e.g. Diffusionsemigroups
- Averaging (Hytonen, Veraar)
  $\int_I \|T(t)\|^r \, dt < \infty$ on X with $1/r > 1/p_X - 1/q_Y$
  $\Rightarrow \left\{\int_I f(t) T(t) \, dt \colon \|f\|_{L^{r'}} \leq 1\right\}$ R-bnd
- Smoothness (Girardi-W., Hytonen-Veraar)
  $T(\cdot) \in B^{d/r}_{r,1}(\mathbb{R}^d, B(X))$ implies R-boundedness for
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Bootstrapping with the Functional Calculus

X has property (α)

- Let A have a $H^\infty(\Sigma_\sigma)$ calculus on X.
  Then $\left\{ f(A) : \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1 \right\}$ is R-bounded.

- Let A have a $\mathcal{H}^{\alpha,2}$ calculus on X.
  Then $\{ f(A) : \|f\|_{\mathcal{H}^{\beta,2}} \leq 1 \}$ is R-bounded for $\beta > \alpha + \frac{1}{2}$.

Kalton, Girardi, Kriegler, W.

Kalton's Factorization Theorem

X Banach space

$\tau \subset B(X)$ γ-bounded, absolutely convex, closed in strong operator top.

Then there exists

- a Hilbert space H and a closed subalgebra $B \subset B(H)$
- $u \colon \widetilde{\tau} = (\mathrm{span}\, \tau, \|\cdot\|_\tau) \to B$ linear, bounded
- $v \colon B \to B(X)$ bounded algebra homomorphism

such that

$$(v \circ u)(T) = T$$

for all $T \in \widetilde{\tau}$.

R-boundedness versus γ-boundedness

X, Y be Banach spaces with $Y \neq \{0\}$

Theorem (Kwapien, Veraar, W.)
Each γ-bounded family $\tau \subset B(X, Y)$ is R-bounded
if and only if X has finite cotype.

Remarks:

- R-boundedness $\Rightarrow$ γ-boundedness, always.
  (γ_n) and $(r_n \gamma_n)$ have the same distribution
- X finite cotype: R-bnd $\iff$ γ-bnd

Proof: "$\Rightarrow$" Assume X not finite cotype. We will construct families $\tau_N \subset B(X, Y)$ with $R^\gamma(\tau_N) / R(\tau_N) \xrightarrow{N \to \infty} 0$.

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Proof of the Theorem Continued

$$\begin{tikzcd} X \arrow[d, "P_N"'] & & & Y \\ X_N \arrow[r, "J_n"] & \ell_N^\infty \arrow[rr, "T"] & & \mathbb{R} \arrow[u, "i"'] \end{tikzcd}$$

$$\|P_N\| \ , \ \|J_N\| \ , \ \left\|J_N^{-1}\right\| \leq C$$

Construct $\tau_N \subset B\left(\ell_N^\infty, \mathbb{R}\right)$ with

(a) $R(\tau_N) \sim N^{1/2}$ (b) $R^\gamma(\tau_N) \sim \left(\frac{N}{\log N}\right)^{1/2}$.

For a fixed N choose $\tau_N = \{T_n((x_1, \ldots, x_N)) = x_n : n = 1, \ldots, N\}$.

Part (a). Let e_n be the unit vectors of ℓ_N^∞.

$$N^{1/2} = \left(\sum_{n=1}^{N} |T_n(e_n)|^2\right)^{1/2} = \mathbb{E}\left|\sum_{n=1}^{N} r_n T_n e_n\right| \leq R(\tau_N)\ \underbrace{\mathbb{E}\left\|\sum_{n=1}^{N} r_n e_n\right\|_{\ell^\infty}}_{=1}$$

Proof of the Theorem Continued

$$\begin{array}{ccccc} X & & & & Y \\ {\scriptstyle P_N}\big\downarrow & & & & \big\uparrow{\scriptstyle i} \\ X_N & \xrightarrow{J_n} & \ell_N^\infty & \xrightarrow{T} & \mathbb{R} \end{array} \qquad \|P_N\| \ , \ \|J_N\| \ , \ \left\|J_N^{-1}\right\| \leq C$$

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Proof Continued

Part (b). For $S_1, \ldots, S_J \in \{T_1, \ldots, T_N\}$ and $x_1, \ldots x_J \in \ell_N^\infty$

$$\left(\mathbb{E}\left|\sum_{j=1}^{J} \gamma_j S_j(x_j)\right|^2\right)^{1/2} \leq 4\left(\frac{N}{\log N}\right)^{1/2}\left(\mathbb{E}\left\|\sum_{j=1}^{J} \gamma_j x_j\right\|^2\right)^{1/2} . \quad (1)$$

$$A_n := \{j : S_j = T_n\} , \; n = 1, \ldots, N .$$

Put $a_n := \left(\sum_{j \in A_n} |T_n(x_j)|^2\right)^{1/2}$ and $\Gamma_n := \sum_{j \in A_n} \gamma_j x_n \in \ell_N^\infty$.

Then (1) takes the form

$$\left(\sum_{n=1}^{N} a_n^2\right)^{1/2} \leq 4\left(\frac{N}{\log N}\right)^{1/2}\left(\mathbb{E}\left\|\sum_{n=1}^{N} \Gamma_n\right\|_\infty^2\right)^{1/2} . \quad (2)$$

For (2) we need for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$

$$\left(\frac{\log N}{N}\sum_{n=1}^{N} \alpha_n^2\right)^{1/2} \leq 4\, \mathbb{E} \sup_{n \leq N} |\alpha_n \gamma_n| \quad (3)$$

Proof Continued

Part (b). For $S_1, \ldots, S_J \in \{T_1, \ldots, T_N\}$ and $x_1, \ldots x_J \in \ell_N^\infty$

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Put $a_n := \left(\sum_{j \in A_n} |T_n(x_j)|^2\right)^{1/2}$ and $\Gamma_n := \sum_{j \in A_n} \gamma_j x_n \in \ell_N^\infty$.

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For (2) we need for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$

$$\left(\frac{\log N}{N}\sum_{n=1}^{N} \alpha_n^2\right)^{1/2} \leq 4\,\mathbb{E}\sup_{n \leq N} |\alpha_n \gamma_n| \quad (3)$$

Proof Continued

Part (b). For $S_1, \ldots, S_J \in \{T_1, \ldots, T_N\}$ and $x_1, \ldots x_J \in \ell_N^\infty$

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Proof (still) Continued

$$\sum_{n=1}^{N} |a_n|^2 \;\le\; \left(\frac{16N}{\log N}\right) \mathbb{E}\sup|\gamma_n a_n|^2 \;=\; \mathbb{E}\sup|\Gamma_n(n)| \tag{4}$$

$(\Gamma_n(n))_{n=1}^N$ and $(\gamma_n a_n)_{n=1}^N$ are Gaussian with the same distribution since

$$\mathbb{E}|\Gamma_n(n)|^2 \;=\; \mathbb{E}\left|\sum_{j\in A_n}\gamma_j T_n(x_j)\right|^2 \;=\; a_n^2 \quad , \quad n=1,\dots N$$

Pointwise on Ω_γ we have

$$\sup_{1\le n\le N}|\Gamma_n(n)| \;=\; \sup_{1\le n\le N}\left|\mathbb{E}_r\left[\sum_{m=1}^{N} r_m r_n(\Gamma_n)(m)\right]\right|^2 \;=\; \left\|\mathbb{E}_r\left[I_n\left(\sum_{m=1}^{N} r_m\Gamma_m\right)\right]\right\|_\infty^2$$

where $I_n(\alpha_k) = (r_n(\cdot)\,\alpha_k)$ is an isometry on ℓ_N^∞

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Continue (4) with this estimate. Since $\Gamma_1,\dots,\Gamma_N$ indep.

$$\sum_{n=1}^{N}|a_n|^2 \le \mathbb{E}_\gamma\mathbb{E}_r\left\|\sum_{m=1}^{N} r_m\Gamma_m\right\|^2 = \mathbb{E}_\gamma\left\|\sum_{m=1}^{n}\Gamma_m\right\|^2 .$$

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R- and γ- boundedness versus R_2-boundedness

Theorem. Let X , Y be Banach lattices. TFAE.

- Every R_2-bounded family $\tau \subset B(X, Y)$ is R-bounded.
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Remarks on R_2-boundedness

X , Y Banach lattices.

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Duality

Notation: For $\tau \subset B(X,Y)$ put $\tau^* = \{T^* : T \in \tau\} \subset B(Y^*, X^*)$.

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Remarks

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