Nonlinear geometry of operator spaces

Bruno de Mendonça Braga
(joint with Thomas Sinclair)

York University

June 20, 2019

Operator spaces

Operator spaces

Let X and Y be vector spaces over the complex numbers and $u : X \to Y$ be a map.

Operator spaces

Let X and Y be vector spaces over the complex numbers and $u : X \to Y$ be a map.

- Given $n \in \mathbb{N}$, the ***n*-th amplification of *u*** is the map $u_n : M_n(X) \to M_n(Y)$ given by $u_n([x_{ij}]) = [u(x_{ij})]$, i.e.,

$$u_n\left(\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix}\right) = \begin{bmatrix} u(x_{11}) & \dots & u(x_{1n}) \\ \vdots & \ddots & \vdots \\ u(x_{n1}) & \dots & u(x_{nn}) \end{bmatrix}.$$

Operator spaces

Let X and Y be vector spaces over the complex numbers and $u : X \to Y$ be a map.

- Given $n \in \mathbb{N}$, the ***n*-th amplification of** u is the map $u_n : M_n(X) \to M_n(Y)$ given by $u_n([x_{ij}]) = [u(x_{ij})]$, i.e.,

$$u_n\left(\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix}\right) = \begin{bmatrix} u(x_{11}) & \dots & u(x_{1n}) \\ \vdots & \ddots & \vdots \\ u(x_{n1}) & \dots & u(x_{nn}) \end{bmatrix}.$$

- Suppose each $M_n(X)$ and $M_n(Y)$ are endowed with norms, say $\|\cdot\|_n$. If u is linear, we call u **completely bounded** if

$$\|u\|_{\mathrm{cb}} = \sup_n \|u_n\| < \infty.$$

Operator spaces

Let X and Y be vector spaces over the complex numbers and $u : X \to Y$ be a map.

- Given $n \in \mathbb{N}$, the ***n*-th amplification of** u is the map $u_n : M_n(X) \to M_n(Y)$ given by $u_n([x_{ij}]) = [u(x_{ij})]$, i.e.,

$$u_n\left(\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix}\right) = \begin{bmatrix} u(x_{11}) & \dots & u(x_{1n}) \\ \vdots & \ddots & \vdots \\ u(x_{n1}) & \dots & u(x_{nn}) \end{bmatrix}.$$

- Suppose each $M_n(X)$ and $M_n(Y)$ are endowed with norms, say $\|\cdot\|_n$. If u is linear, we call u **completely bounded** if

$$\|u\|_{\mathrm{cb}} = \sup_n \|u_n\| < \infty.$$

- We call u a **complete isomorphism** if it is linear and both u and u^{-1} are completely bounded.

Operator spaces

Operator spaces

Let H be a Hilbert space and X be a closed subspace of $\mathcal{B}(H)$.

Operator spaces

Let H be a Hilbert space and X be a closed subspace of $\mathcal{B}(H)$.

- Given $n \in \mathbb{N}$, $M_n(\mathcal{B}(H))$ has a natural Banach space structure given by the canonical isomorphism between $M_n(\mathcal{B}(H))$ and $\mathcal{B}(H^{\oplus n})$, where

$$H^{\oplus n} = \underbrace{H \oplus_2 \ldots \oplus_2 H}_{n\text{-times}}.$$

Operator spaces

Let H be a Hilbert space and X be a closed subspace of $\mathcal{B}(H)$.

- Given $n \in \mathbb{N}$, $M_n(\mathcal{B}(H))$ has a natural Banach space structure given by the canonical isomorphism between $M_n(\mathcal{B}(H))$ and $\mathcal{B}(H^{\oplus n})$, where
$$H^{\oplus n} = \underbrace{H \oplus_2 \ldots \oplus_2 H}_{n\text{-times}}.$$
- For each $n \in \mathbb{N}$, we consider $M_n(X)$ endowed the norm $\|\cdot\|_n$ given by the inclusion $M_n(X) \subset M_n(\mathcal{B}(H))$.

Operator spaces

Let H be a Hilbert space and X be a closed subspace of $\mathcal{B}(H)$.

- Given $n \in \mathbb{N}$, $M_n(\mathcal{B}(H))$ has a natural Banach space structure given by the canonical isomorphism between $M_n(\mathcal{B}(H))$ and $\mathcal{B}(H^{\oplus n})$, where

$$H^{\oplus n} = \underbrace{H \oplus_2 \ldots \oplus_2 H}_{n\text{-times}}.$$

- For each $n \in \mathbb{N}$, we consider $M_n(X)$ endowed the norm $\|\cdot\|_n$ given by the inclusion $M_n(X) \subset M_n(\mathcal{B}(H))$.

Definition

An **operator space** is a closed subspace of $\mathcal{B}(H)$, for some Hilbert space H, and the morphisms between operator spaces are the completely bounded maps.

Operator spaces

Operator spaces

- Alternatively, let X be a vector space and $(\|\cdot\|_n)_n$ be norms on $(M_n(X))_n$. Consider the following.

Operator spaces

- Alternatively, let X be a vector space and $(\|\cdot\|_n)_n$ be norms on $(M_n(X))_n$. Consider the following.

  (R1) $\|\alpha x\beta\|_n \leqslant \|\alpha\|\|x\|_n\|\beta\|$ for all $\alpha, \beta \in M_n(\mathbb{C})$ and all $x \in M_n(X)$.

Operator spaces

- Alternatively, let X be a vector space and $(\|\cdot\|_n)_n$ be norms on $(M_n(X))_n$. Consider the following.

(R1) $\|\alpha x\beta\|_n \leqslant \|\alpha\|\|x\|_n\|\beta\|$ for all $\alpha, \beta \in M_n(\mathbb{C})$ and all $x \in M_n(X)$.
(R2) For all $n, m \in \mathbb{N}$, all $x \in M_n(X)$ and all $y \in M_m(X)$, we have

$$\left\|\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}.$$

Operator spaces

- Alternatively, let X be a vector space and $(\|\cdot\|_n)_n$ be norms on $(M_n(X))_n$. Consider the following.

  (R1) $\|\alpha x\beta\|_n \leqslant \|\alpha\|\|x\|_n\|\beta\|$ for all $\alpha, \beta \in M_n(\mathbb{C})$ and all $x \in M_n(X)$.
  (R2) For all $n, m \in \mathbb{N}$, all $x \in M_n(X)$ and all $y \in M_m(X)$, we have

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}.$$

- Clearly, if X is an operator space, the sequence $(\|\cdot\|_n)_n$ of norms on $(M_n(X))_n$ given by the inclusion $X \subset \mathcal{B}(H)$ satisfies (R1) and (R2).

Operator spaces

- Alternatively, let X be a vector space and $(\|\cdot\|_n)_n$ be norms on $(M_n(X))_n$. Consider the following.

  (R1) $\|\alpha x \beta\|_n \leqslant \|\alpha\|\|x\|_n\|\beta\|$ for all $\alpha, \beta \in M_n(\mathbb{C})$ and all $x \in M_n(X)$.
  (R2) For all $n, m \in \mathbb{N}$, all $x \in M_n(X)$ and all $y \in M_m(X)$, we have

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}.$$

- Clearly, if X is an operator space, the sequence $(\|\cdot\|_n)_n$ of norms on $(M_n(X))_n$ given by the inclusion $X \subset \mathcal{B}(H)$ satisfies (R1) and (R2).

Theorem (Z.-J. Ruan, 1988)

Let X be a vector space and $(\|\cdot\|_n)_n$ be a sequence of norms on $(M_n(X))_n$. Then X is completely isometrically isomorphic to a subspace of $\mathcal{B}(H)$, for some Hilbert space H, if and only if $(\|\cdot\|_n)_n$ satisfies (R1) and (R2).

Examples

Examples

- Let X be a Banach space and consider its canonical embedding into the abelian C^*-algebra $C(B_{X*})$. The canonical isomorphim between $M_n(C(B_{X*}))$ and $C(B_{X*}, M_n(\mathbb{C}))$ gives us an operator space structure on X.

Examples

- Let X be a Banach space and consider its canonical embedding into the abelian C^*-algebra $C(B_{X*})$. The canonical isomorphim between $M_n(C(B_{X*}))$ and $C(B_{X*}, M_n(\mathbb{C}))$ gives us an operator space structure on X.
- C^*-algebras have unique operator space structures.

Examples

- Let X be a Banach space and consider its canonical embedding into the abelian C^*-algebra $C(B_{X*})$. The canonical isomorphim between $M_n(C(B_{X*}))$ and $C(B_{X*}, M_n(\mathbb{C}))$ gives us an operator space structure on X.
- C^*-algebras have unique operator space structures.
- The Hilbert space ℓ_2 is isometric to both

$$R = \{a \in \mathcal{B}(\ell_2) \mid a_{ij} \neq 0 \Rightarrow i = 1\}$$

Examples

- Let X be a Banach space and consider its canonical embedding into the abelian C^*-algebra $C(B_{X^*})$. The canonical isomorphim between $M_n(C(B_{X^*}))$ and $C(B_{X^*}, M_n(\mathbb{C}))$ gives us an operator space structure on X.
- C^*-algebras have unique operator space structures.
- The Hilbert space ℓ_2 is isometric to both

$$R = \{a \in \mathcal{B}(\ell_2) \mid a_{ij} \neq 0 \Rightarrow i = 1\}$$

and

$$C = \{a \in \mathcal{B}(\ell_2) \mid a_{ij} \neq 0 \Rightarrow j = 1\}.$$

Examples

- Let X be a Banach space and consider its canonical embedding into the abelian C^*-algebra $C(B_{X*})$. The canonical isomorphim between $M_n(C(B_{X*}))$ and $C(B_{X*}, M_n(\mathbb{C}))$ gives us an operator space structure on X.
- C^*-algebras have unique operator space structures.
- The Hilbert space ℓ_2 is isometric to both

$$R = \{a \in \mathcal{B}(\ell_2) \mid a_{ij} \neq 0 \Rightarrow i = 1\}$$

and

$$C = \{a \in \mathcal{B}(\ell_2) \mid a_{ij} \neq 0 \Rightarrow j = 1\}.$$

Proposition

Let X be an operator space and A be an abelian C^*-algebra. Then every bounded linear map $u : X \to A$ satisfies $\|u\| = \|u\|_{\mathrm{cb}}$.

Basic definitions on nonlinear geometry

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u : X \to Y$ be a map.

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u : X \to Y$ be a map.

1. The **modulus of uniform continuity of** u is given by

$$\omega_u(t) = \sup\Big\{\|u(x) - u(y)\| \mid \|x - y\| \leqslant t\Big\}.$$

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u : X \to Y$ be a map.

1. The **modulus of uniform continuity of** u is given by

$$\omega_u(t) = \sup \Big\{ \|u(x) - u(y)\| \mid \|x - y\| \leqslant t \Big\}.$$

2. The **complete modulus of uniform continuity of** u is given by

$$\omega_u^{\mathrm{cb}}(t) = \sup_n \omega_{u_n}(t).$$

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u : X \to Y$ be a map.

1. The **modulus of uniform continuity of** u is given by

$$\omega_u(t) = \sup \Big\{ \|u(x) - u(y)\| \mid \|x - y\| \leqslant t \Big\}.$$

2. The **complete modulus of uniform continuity of** u is given by

$$\omega_u^{\mathrm{cb}}(t) = \sup_n \omega_{u_n}(t).$$

3. u is **completely coarse** if $\omega_u^{\mathrm{cb}}(t) < \infty$ for all $t > 0$.

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u : X \to Y$ be a map.

1. The **modulus of uniform continuity of** u is given by

$$\omega_u(t) = \sup\Big\{\|u(x) - u(y)\| \mid \|x - y\| \leqslant t\Big\}.$$

2. The **complete modulus of uniform continuity of** u is given by

$$\omega_u^{\mathrm{cb}}(t) = \sup_n \omega_{u_n}(t).$$

3. u is **completely coarse** if $\omega_u^{\mathrm{cb}}(t) < \infty$ for all $t > 0$.
4. u is **completely uniformly continuous** if $\lim_{t\to 0^+} \omega_u^{\mathrm{cb}}(t) = 0$.

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u : X \to Y$ be a map.

1. The **modulus of uniform continuity of** u is given by

$$\omega_u(t) = \sup\Big\{\|u(x) - u(y)\| \mid \|x - y\| \leqslant t\Big\}.$$

2. The **complete modulus of uniform continuity of** u is given by

$$\omega_u^{\mathrm{cb}}(t) = \sup_n \omega_{u_n}(t).$$

3. u is **completely coarse** if $\omega_u^{\mathrm{cb}}(t) < \infty$ for all $t > 0$.
4. u is **completely uniformly continuous** if $\lim_{t\to 0^+} \omega_u^{\mathrm{cb}}(t) = 0$.
5. u is **completely Lipschitz** if $\mathrm{Lip}_{\mathrm{cb}}(u) = \sup_{t>0} \omega_u^{\mathrm{cb}}(t)/t < \infty$.

Basic definitions on nonlinear geometry

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u, w : X \to Y$ be maps.

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u, w : X \to Y$ be maps.

1. u is a **complete uniform equivalence** if it is a completely uniformly continuous bijection and u^{-1} is completely uniformly continuous.

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u, w : X \to Y$ be maps.

1. u is a **complete uniform equivalence** if it is a completely uniformly continuous bijection and u^{-1} is completely uniformly continuous.
2. u is a **complete Lipschitz equivalence** if it is a completely Lipschitz bijection and u^{-1} is completely Lipschitz.

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u, w : X \to Y$ be maps.

1. u is a **complete uniform equivalence** if it is a completely uniformly continuous bijection and u^{-1} is completely uniformly continuous.
2. u is a **complete Lipschitz equivalence** if it is a completely Lipschitz bijection and u^{-1} is completely Lipschitz.
3. The maps u and w are **completely close**, write $u \sim w$, if

$$\sup_n \sup_{x \in M_n(X)} \|u_n(x) - v_n(x)\| < \infty.$$

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u, w : X \to Y$ be maps.

1. u is a **complete uniform equivalence** if it is a completely uniformly continuous bijection and u^{-1} is completely uniformly continuous.
2. u is a **complete Lipschitz equivalence** if it is a completely Lipschitz bijection and u^{-1} is completely Lipschitz.
3. The maps u and w are **completely close**, write $u \sim w$, if

$$\sup_n \sup_{x \in M_n(X)} \|u_n(x) - v_n(x)\| < \infty.$$

4. u is a **complete coarse equivalence** if it is completely coarse and there is a completely coarse $v : Y \to X$ such that $v \circ u \sim \mathrm{Id}_X$ and $u \circ v \sim \mathrm{Id}_Y$.

Basic definitions on nonlinear geometry

Definition

Let X and Y be operator spaces and $u, w : X \to Y$ be maps.

1. u is a **complete uniform equivalence** if it is a completely uniformly continuous bijection and u^{-1} is completely uniformly continuous.
2. u is a **complete Lipschitz equivalence** if it is a completely Lipschitz bijection and u^{-1} is completely Lipschitz.
3. The maps u and w are **completely close**, write $u \sim w$, if
$$\sup_n \sup_{x \in M_n(X)} \|u_n(x) - v_n(x)\| < \infty.$$
4. u is a **complete coarse equivalence** if it is completely coarse and there is a completely coarse $v : Y \to X$ such that $v \circ u \sim \mathrm{Id}_X$ and $u \circ v \sim \mathrm{Id}_Y$.

Which theorems on the nonlinear theory of Banach spaces have an operator space version?

Differentiability

Differentiability

Theorem (Heinrich and Mankiewicz, 1982)

Let X and Y be Banach spaces and X be separable. If X Lipschitzly embeds into Y^*, then X isomorphically embeds into Y^*.

Differentiability

Theorem (Heinrich and Mankiewicz, 1982)

Let X and Y be Banach spaces and X be separable. If X Lipschitzly embeds into Y^*, then X isomorphically embeds into Y^*.

Recall: A map $f : X \to Y^*$ is **Gateaux w^*-differentiable at** $x \in X$ if for all $a \in X$ the limit

$$D^* f_x(a) = w^*\text{-}\lim_{\lambda \to 0} \frac{f(x + \lambda a) - f(x)}{\lambda}$$

exists and the map $D^* f_x : a \in X \mapsto D^* f_x(a) \in Y^*$ is linear and bounded.

Differentiability

Theorem (Heinrich and Mankiewicz, 1982)

Let X and Y be Banach spaces and X be separable. If X Lipschitzly embeds into Y^*, then X isomorphically embeds into Y^*.

Recall: A map $f : X \to Y^*$ is **Gateaux w^*-differentiable at** $x \in X$ if for all $a \in X$ the limit

$$D^* f_x(a) = w^*\text{-}\lim_{\lambda \to 0} \frac{f(x + \lambda a) - f(x)}{\lambda}$$

exists and the map $D^* f_x : a \in X \mapsto D^* f_x(a) \in Y^*$ is linear and bounded.

If X and Y are separable, a Lipschitz function $u : X \to Y^*$ is Gateaux w^*-differentiable "almost everywhere" **(Heinrich and Mankiewicz)**.

Differentiability

Differentiability

- If $D^* f_x(a)$ exists, lower w^*-continuity of the norm easily gives $\|D^* f_x(a)\| \leqslant \mathrm{Lip}(f)$ and $\|D^* f_x(a)\|_{\mathrm{cb}} \leqslant \mathrm{Lip}_{\mathrm{cb}}(f)$.

Differentiability

- If $D^* f_x(a)$ exists, lower w^*-continuity of the norm easily gives $\|D^* f_x(a)\| \leqslant \mathrm{Lip}(f)$ and $\|D^* f_x(a)\|_{\mathrm{cb}} \leqslant \mathrm{Lip}_{\mathrm{cb}}(f)$.
- The difficulty is to show that $\|(D^* f_x(a))^{-1}\| \leqslant \mathrm{Lip}(f^{-1})$.

Differentiability

- If $D^* f_x(a)$ exists, lower w^*-continuity of the norm easily gives $\|D^* f_x(a)\| \leqslant \mathrm{Lip}(f)$ and $\|D^* f_x(a)\|_{\mathrm{cb}} \leqslant \mathrm{Lip}_{\mathrm{cb}}(f)$.
- The difficulty is to show that $\|(D^* f_x(a))^{-1}\| \leqslant \mathrm{Lip}(f^{-1})$.
- This is also the part of the proof that does not straightforwardly extends to operator spaces.

Differentiability

- If $D^* f_x(a)$ exists, lower w^*-continuity of the norm easily gives $\|D^* f_x(a)\| \leqslant \mathrm{Lip}(f)$ and $\|D^* f_x(a)\|_{\mathrm{cb}} \leqslant \mathrm{Lip}_{\mathrm{cb}}(f)$.
- The difficulty is to show that $\|(D^* f_x(a))^{-1}\| \leqslant \mathrm{Lip}(f^{-1})$.
- This is also the part of the proof that does not straightforwardly extends to operator spaces.
- For operator spaces, its necessary to find a “nice” subset $F \subset X$ for which $x \in F \mapsto D^* f_x(a) \in Y^*$ is continuous.

Differentiability

- If $D^* f_x(a)$ exists, lower w^*-continuity of the norm easily gives $\|D^* f_x(a)\| \leqslant \mathrm{Lip}(f)$ and $\|D^* f_x(a)\|_{\mathrm{cb}} \leqslant \mathrm{Lip}_{\mathrm{cb}}(f)$.
- The difficulty is to show that $\|(D^* f_x(a))^{-1}\| \leqslant \mathrm{Lip}(f^{-1})$.
- This is also the part of the proof that does not straightforwardly extends to operator spaces.
- For operator spaces, its necessary to find a "nice" subset $F \subset X$ for which $x \in F \mapsto D^* f_x(a) \in Y^*$ is continuous.

Theorem

Let X and Y be operator spaces and X be separable. If X completely Lipschitzly embeds into Y^*, then X completely isomorphically embeds into Y^*.

Ribe's Theorem

Ribe's Theorem

A Banach (resp. operator) space X is (resp. **completely**) **crudely finitely representable** in a Banach (resp. operator) space Y if there is $\lambda \geqslant 1$ so that every finite dimensional $F \subset X$ is λ-isomorphic (resp. λ-completely isomorphic) to some $E \subset Y$.

Ribe's Theorem

A Banach (resp. operator) space X is (resp. **completely**) **crudely finitely representable** in a Banach (resp. operator) space Y if there is $\lambda \geqslant 1$ so that every finite dimensional $F \subset X$ is λ-isomorphic (resp. λ-completely isomorphic) to some $E \subset Y$.

Theorem (Ribe, 1976)

Let X and Y be uniformly (or coarsely) equivalent Banach spaces, then X and Y are crudely finitely representable in each other.

Ribe's Theorem

A Banach (resp. operator) space X is (resp. **completely**) **crudely finitely representable** in a Banach (resp. operator) space Y if there is $\lambda \geqslant 1$ so that every finite dimensional $F \subset X$ is λ-isomorphic (resp. λ-completely isomorphic) to some $E \subset Y$.

Theorem (Ribe, 1976)

Let X and Y be uniformly (or coarsely) equivalent Banach spaces, then X and Y are crudely finitely representable in each other.

Things are not as simple for operator spaces due to the possible lack of local reflexivity.

Ribe's Theorem

A Banach (resp. operator) space X is (resp. **completely**) **crudely finitely representable** in a Banach (resp. operator) space Y if there is $\lambda \geqslant 1$ so that every finite dimensional $F \subset X$ is λ-isomorphic (resp. λ-completely isomorphic) to some $E \subset Y$.

Theorem (Ribe, 1976)

Let X and Y be uniformly (or coarsely) equivalent Banach spaces, then X and Y are crudely finitely representable in each other.

Things are not as simple for operator spaces due to the possible lack of local reflexivity.

Let X be an operator space, X^* has a natural operator space structure. X is **locally reflexive** if X^{**} is completely finitely representable in X.

Ribe's Theorem

A Banach (resp. operator) space X is (resp. **completely**) **crudely finitely representable** in a Banach (resp. operator) space Y if there is $\lambda \geqslant 1$ so that every finite dimensional $F \subset X$ is λ-isomorphic (resp. λ-completely isomorphic) to some $E \subset Y$.

Theorem (Ribe, 1976)

Let X and Y be uniformly (or coarsely) equivalent Banach spaces, then X and Y are crudely finitely representable in each other.

Things are not as simple for operator spaces due to the possible lack of local reflexivity.

Let X be an operator space, X^* has a natural operator space structure. X is **locally reflexive** if X^{**} is completely finitely representable in X.

$\mathcal{K}(\ell_2)$ is not locally reflexive **(Effros, Junge and Ruan)**.

Ribe's Theorem

Ribe's Theorem

A C^*-subalgebra of $\mathcal{B}(H)$ which is closed with respect to the strong operator topology is called a **von Neumann algebra.**

Ribe's Theorem

A C^*-subalgebra of $\mathcal{B}(H)$ which is closed with respect to the strong operator topology is called a **von Neumann algebra**.

Every von Neumann algebra is the dual of an operator space.

Ribe's Theorem

A C^*-subalgebra of $\mathcal{B}(H)$ which is closed with respect to the strong operator topology is called a **von Neumann algebra**.

Every von Neumann algebra is the dual of an operator space.

Theorem (Effros, Junge and Ruan, 2000)

Preduals of von Neumann algebras are locally reflexive.

Ribe's Theorem

A C^*-subalgebra of $\mathcal{B}(H)$ which is closed with respect to the strong operator topology is called a **von Neumann algebra**.

Every von Neumann algebra is the dual of an operator space.

Theorem (Effros, Junge and Ruan, 2000)

Preduals of von Neumann algebras are locally reflexive.

Theorem

Let X and Y be completely coarsely equivalent preduals of von Neumann algebras, then X and Y are completely crudely finitely representable in each other.

The operator space OH

The operator space OH

Theorem (Pisier, 1996)

There exists an operator space OH *so that*

The operator space OH

Theorem (Pisier, 1996)

There exists an operator space OH *so that*

1. OH *is isometric as a Banach space to* ℓ_2*, and*

The operator space OH

Theorem (Pisier, 1996)

There exists an operator space OH *so that*

1. OH *is isometric as a Banach space to* ℓ_2*, and*
2. *the canonical identification between* OH *and* $\overline{\mathrm{OH}^*}$*, coming from the canonical identification between* ℓ_2 *and* $\overline{\ell_2^*}$*, is a complete isometry.*

The operator space OH

Theorem (Pisier, 1996)

There exists an operator space OH *so that*

1. OH *is isometric as a Banach space to* ℓ_2*, and*
2. *the canonical identification between* OH *and* $\overline{\mathrm{OH}^*}$*, coming from the canonical identification between* ℓ_2 *and* $\overline{\ell_2^*}$*, is a complete isometry.*

   Moreover, OH *is the unique space, up to complete isometry, satisfying the properties above.*

The operator space OH

Theorem (Pisier, 1996)

There exists an operator space OH *so that*

1. OH *is isometric as a Banach space to* ℓ_2*, and*
2. *the canonical identification between* OH *and* $\overline{\mathrm{OH}^*}$*, coming from the canonical identification between* ℓ_2 *and* $\overline{\ell_2^*}$*, is a complete isometry.*

   Moreover, OH *is the unique space, up to complete isometry, satisfying the properties above.*

Theorem

If an operator space is completely coarsely equivalent to OH*, then it is completely isomorphic to* OH*.*

Merci!