TWISTING HILBERT SPACES, EXPLAINED BY THEMSELVES

JESÚS M. F. CASTILLO

1. A Hilbert space is a complete normed space whose norm $\|\cdot\|$ comes induced by an inner product $(\cdot,\cdot)$ in the form $\|x\| = (x,x)^{1/2}$. The orthogonal projection (which should not be linear, but it is) provides a contractive projection onto every closed subspace.

2. Which Hilbert space is then one to consider: $\ell_2, L_2(0,1), L_2(\mathbb{R})$, the Schatten class S_2, ... ? The fact that all of them are isometric is just an outstanding theorem. So, better let us consider them as different spaces.

3. The Eilenberg-McLane program [9, 6] establishes as a foundational line that only categories and functors are objects of study in mathematics.

4. Complex interpolation is a natural place were the Eilenberg-McLane is subtly demoed: by the virtues of classical Riesz-Thorin theorem, when a linear operator $\ell_\infty \longrightarrow \ell_\infty$ also acts continuously $\ell_1 \longrightarrow \ell_1$ it automatically acts continuously from $\ell_2 \longrightarrow \ell_2$. ✠

5. There is a way to say that: those Hilbert spaces live in different categories, even if, as mere Banach spaces, all of them are isometric.

- ℓ_2 is an ℓ_∞-Banach module,
- $L_2(0,1)$ is an $L_\infty(0,1)$-Banach module,
- $L_2(\mathbb{R})$ is an $L_\infty(\mathbb{R})$-Banach module,
- S_2 is an $\mathfrak{L}(\ell_2,\ell_2)$-Banach module,
- etc.

6. It was Kalton's discovery that exact sequences of quasi-Banach spaces (in general), and of Hilbert spaces (in particular) can be described by using a certain type of nonlinear maps, called quasi-linear maps.

$$0 \longrightarrow \ell_2 \longrightarrow Z \longrightarrow \ell_2 \longrightarrow 0$$

7. Quasi-linear maps (?), Centralizers (?) or Derivations (also called differentials) (?) are three type of maps $\Omega : \ell_2 \to \Sigma$, all of them produce the so-called derived space Z in the form

$$\begin{aligned} Z &= \ell_2 \oplus_\Omega \ell_2 \\ &= \{(w, x) \in \Sigma \times \ell_2 : w - \Omega(x) \in \ell_2\} \\ &= \{(f'(\theta), f(\theta)) : f \in \mathcal{H}\} \end{aligned}$$

endowed with the quasi-norm/norm

$$\begin{aligned} \|(w, x)\|_\Omega &= \|w - \Omega(x)\| + \|x\| \\ &\sim \inf\{\|f\| : f'(\theta) = w, \quad f(\theta) = x\}. \end{aligned}$$

8. The derived space Z is actually in a twisted Hilbert space; i.e., the middle space in an exact sequence

$$0 \longrightarrow \ell_2 \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } \ell_2 \longrightarrow 0$$

whose meaning is:

- ℓ_2 is a subspace of Z through the isomorphic embedding $j(y) = (y, 0)$.
- and ℓ_2 is the corresponding quotient $Z/j(Y)$ through the quotient map $q(w, x) = x$.

9. Derivations appear naturally in complex interpolation. Whenever $X_\theta = (X_0, X_1)_\theta$ is an interpolation space, there is a derivation Ω_θ on X_θ that, therefore, produces an exact sequence

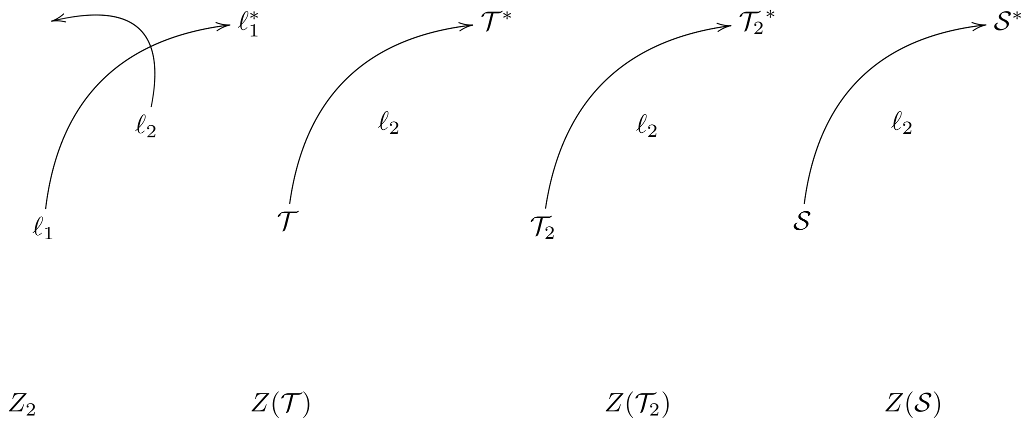
$$0 \longrightarrow X_\theta \longrightarrow Z \longrightarrow X_\theta \longrightarrow 0 \qquad (\Omega_\theta : X_\theta \curvearrowright X_\theta)$$

10. On the other hand, it is a rather simple thing to get $\ell_2 = (X, X^*)_{1/2}$.

11. So, name the space X and a twisted Hilbert appears

$$0 \longrightarrow \ell_2 \longrightarrow Z(X) \longrightarrow \ell_2 \longrightarrow 0 \qquad (\Omega : \ell_2 \curvearrowright \ell_2)$$

12. Say



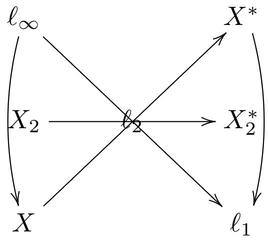
13. Kalton's outstanding theorem says that, in the domain of Köthe spaces, the two things, derivations and interpolation scales are the same.
Name the Ω on ℓ_2 and I will tell you from which X you got $\ell_2 \oplus_\Omega \ell_2 = Z(X)$.

14. The scale of ℓ_p-spaces produces the Kalton-Peck space Z_2 and the derivation $\mathcal{K}(x) = x \log x$.

15. We are thus ready to face the problem of the Classification of Twisted Hilbert spaces.
- Is every twisted Hilbert space isomorphic to its dual?
- Is every twisted Hilbert space isomorphic to its square?
- Is a twisted Hilbert space isomorphic to its hyperplanes?
- Do there exist non-ergodic (nontrivial) twisted Hilbert spaces?
- Do there exist prime twisted Hilbert spaces?

Some properties that currently classify Twisted Hilbert spaces: strict singularity, W_2, to be weak-Hilbert, to be asymptotically Hilbert, to be liquid, to be solid.

16. The space Z_2 and the derivation $\mathcal{K}$ pervade the whole theory.
For instance



yields

$$\Omega_{X_2} = \frac{1}{2}\left(\Omega_X + \mathcal{K}\right)$$

17. The Kalton-Peck map

$$\mathcal{K}(x) = x \log |x|$$

appears as soon as it can. For functions

$$\mathcal{K}(f) = f \log |f|$$

For operators

$$\Omega(\tau) = \tau \log |\tau|$$

...

18. For non-commutative L_p-spaces [3], [1], for Operator Spaces [8]...

19. Indeed, the Kalton-Peck map has a stubborn determination to be a centralizer on any category in which one can barely make a sense of it. Consequently, each of those categories has:

- Its own Hilbert space,
- Its own twisted Hilbert Z_2 space

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Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas, 06071-Badajoz, Spain
E-mail address: `castillo@unex.es`