About complementably universal and ergodic Banach spaces

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Introduction

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$(\sum \oplus \ell_r^n)_p$ is not isomorphic to ℓ_p, $(1 \leq p < r < 2)$.

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Investigate the complexity of the isomorphism relation between subspaces of a Banach space non-isomorphic to ℓ_2

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$\mathcal{SB}$, when $X = C(2^{\mathbb{N}})$.

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- $|X/E| \leq |Y/F|$
- E_0 defined on $2^{\mathbb{N}}$

$$xE_0y \iff \exists N\, \forall n > N,\ x(n) = y(n).$$

Ergodic spaces

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- *Ferenczi-Galego (2006) c_0 and ℓ_p $(1 \leq p < 2)$ are ergodic.*
- *Question: Is ℓ_p $(2 < p < \infty)$ ergodic?*

Ergodic spaces

Ferenczi-Rosendal (2005) If X is non-ergodic with unconditional basis, then X is isomorphic to

- Its hiperplanes.
- X^2.
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Conjecture: Ferenczi-Rosendal Every separable Banach space not isomorphic to ℓ_2 is ergodic.

Main results

Theorem

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A separable non-ergodic Banach space must be near Hilbert.

X is said to be near Hilbert when $p(X) = q(X) = 2$, when

$$\begin{aligned} p(X) &= \sup\{p : X \text{ has type } p\}, \\ q(X) &= \inf\{q : X \text{ has cotype } q\}. \end{aligned}$$

Approximation property

Definition

A Banach space X is said to have A.P. if for every compact $K \subseteq X$ and every $\epsilon > 0$, there exists a finite rank operator $T : X \to X$ such that $\|Tz - z\| < \epsilon$ for every $z \in K$.

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(ii) For every linear and bounded operator $T : X \to X$ and $n \in \mathbb{N}$ we have

$$|\beta_n(T) - \beta_{n-1}(T)| \leq \alpha_n \|T\|$$

with $\sum_n \alpha_n < \infty$, where

$$\beta_n(T) = \frac{1}{2^n} \sum_{2^n \leq k < 2^{n+1}} x_k^*(Tx_k)$$

Cantorized-Enflo criterion

Consider $I_n = \{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\} = I_{n,0} \cup I_{n,1}$.

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Given a sequence $\{x_k\}_k \subset X$ and $t \in 2^{\mathbb{N}}$, we write

$$X_t = \overline{span}\{x_k \ : k \in I_{n,t(n)},\, n = 2, 3, \dots\}$$

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(ii) $\sum_n \alpha_n < \infty$.

Examples

Theorem (Johnson-Szankowski, 1976)

ℓ_p, *($2 < p < \infty$) satisfies Cantorized-Enflo criterion.*

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If ℓ_p $(1 \leq p < \infty,\ p \neq 2)$ is finitely representable in a Banach space X, then X satisfies E-C criterion.

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Maurey-Pisier; Krivine For every Banach space X, $\ell_{p(X)}$ and $\ell_{q(X)}$ are finitely representable in X.

C-E criterion and Complementably universal

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Proposition

Let X be a Banach space satisfying the C-E criterion and let $\Gamma \subseteq 2^{\mathbb{N}}$ uncountable. Then there is no separable Banach space which is complementably universal for $\{X_t,\, t \in \Gamma\}$.

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The function $\phi : 2^{\mathbb{N}} \to \mathcal{SB}(X)$ given by $\phi(t) = X_t$ is Borel.

Problems

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Is every non-trivial twisted Hilbert space ergodic?