Finitely additive measures and complementability of Lipschitz-free spaces

Marek Cúth

Septiemes Journees Besancon-Neuchatel d'analyse fonctionnelle

References

M. Cúth, O. Kalenda and P. Kaplický, *Finitely additive measures and complementability of Lipschitz-free spaces*, preprint available at arxiv.org

1. Lipschitz-free spaces in general and motivation to our result
   - Definition and universal property
   - Motivation and main result
2. Characterization of $\mathcal{F}(\mathbb{R}^d)$
3. Sketch of the proof of our main theorem

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- $\mathcal{F}([0,\infty)) \equiv L^1([0,\infty))$and $\delta(x) = \chi_{[0,x)} \in L^1$ for $x > 0$.

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Theorem (Main result)

Let E be a normed linear space of a finite dimension $d \geq 2$. Then there is a linear projection $Q : \mathcal{F}(E)^{**} \to \mathcal{F}(E)$ such that $\|Q\| \leq d_{BM}(E, \ell_2^d)$.

Representation of $\mathcal{F}(\mathbb{R}^d)$

Proposition

- *The mapping* $T : f \mapsto f'$ *is a linear isometry of* $\mathrm{Lip}_0(\mathbb{R}^d)$ *into* $L^\infty(\mathbb{R}^d, \mathbb{R}^d)$. *The range of* T *is*

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$$X(\mathbb{R}^d) = \{\boldsymbol{f} = (f_1, \dots, f_d) \in L^\infty(\mathbb{R}^d, \mathbb{R}^d) :\ \forall i, j \leq d : \partial_i f_j = \partial_j f_i \text{ in } \mathcal{D}'(\mathbb{R}^d)\}$$

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- $X(\mathbb{R}^d) = Y(\mathbb{R}^d)^\perp$*, where*

$$Y(\mathbb{R}^d) = \{\boldsymbol{g} \in L^1(\mathbb{R}^d, \mathbb{R}^d) :\ \operatorname{div} \boldsymbol{g} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d)\}$$

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- *The adjoint mapping* T^* *maps the quotient* $L^1(\mathbb{R}^d, \mathbb{R}^d)/Y(\mathbb{R}^d)$ *(which is the canonical predual of* $X(\mathbb{R}^d)$*) isometrically onto* $\mathcal{F}(\mathbb{R}^d)$.

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- *Moreover, if* $\boldsymbol{a} \in \mathbb{R}^d$ *is arbitrary, then in this identification we have*

  $$\delta(\boldsymbol{a}) = [\boldsymbol{g}] \text{ if and only if } \boldsymbol{g} \in L^1(\mathbb{R}^d, \mathbb{R}^d) \text{ and } \operatorname{div} \boldsymbol{g} = \varepsilon_{\boldsymbol{o}} - \varepsilon_{\boldsymbol{a}} \text{ in } \mathcal{D}'(\mathbb{R}^d),$$

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Our approach: For a nonnegative measure $\nu \in \mathcal{M}_\sigma(\mathbb{R}^d)$, we set

$$\mathcal{M}_\nu(E) = \{\boldsymbol{f} \in L^1(\nu, \mathbb{R}^d) : \ \boldsymbol{f}\nu \in \mathcal{M}_\sigma^{div}(\mathbb{R}^d)\}.$$

Proposition

Let $\nu \in \mathcal{M}_\sigma(E)$ be a nonnegative measure. Then there exists a mapping T_ν assigning to each $\boldsymbol{x} \in \mathbb{R}^d$ a vector subspace $T_\nu(\boldsymbol{x}) \subset E$ with the following properties:

- *T_ν is lower ν-measurable, i.e. $\{\boldsymbol{x} \in \mathbb{R}^d : \ T_\nu(\boldsymbol{x}) \cap G \neq \emptyset\}$ is ν-measurable for any $G \subset E$ open.*
- *For any $\boldsymbol{f} \in L^1(\nu, E)$ we have*

$$\boldsymbol{f} \in \mathcal{M}_\nu(E) \Leftrightarrow \boldsymbol{f}(\boldsymbol{x}) \in T_\nu(\boldsymbol{x}) \text{ for } \nu\text{-almost all } \boldsymbol{x} \in \mathbb{R}^d.$$

Moreover, there exists a sequence $(\boldsymbol{f}_n)_{n=1}^\infty$ of functions from $\mathcal{M}_\nu(E)$ such that

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The projection we need is then defined as "$\boldsymbol{f}|\boldsymbol{\mu}| \mapsto (x \mapsto P_{T_{|\boldsymbol{\mu}|}(x)}\boldsymbol{f}(x))|\boldsymbol{\mu}|$"