Hilbert-avoiding dichotomies and generalizations of the homogeneous space problem.

Wilson Cuellar[1]
Noé de Rancourt[2]
Valentin Ferenczi[1]

[1]Universidade de São Paulo, IME
[2]Université Paris 7, IMJ-PRG

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Question (Banach's homogeneous space problem, 1932)

Say that a space is homogeneous if it is isomorphic to all of its subspaces. Is every homogeneous space Hilbertian ?

The answer is yes ; it relies on three results. Recall that a space X is hereditarily indecomposable (HI) if no two subspaces of X are in topological direct sum.

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Theorem (Gowers' first dichotomy, 1995)

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Conjecture (Ferenczi – Rosendal, 2004)

Every non-Hilbertian space has continuum-many pairwise non-isomorphic subspaces.

Ferenczi–Rosendal's conjecture is actually stronger: they conjecture that if X is non-Hilbertian, then the equivalence relation $\mathbf{E_0}$ Borel-reduces to the isomorphism relation between subspaces of X.

Ferenczi–Rosendal's conjecture

The last conjecture is supported by several regularity results for spaces with less than continuum-many subspaces, proved by Ferenczi and Rosendal (2005). Let X be such a space.

(1) If X has an unconditional basis, then:

- X is isomorphic to its hyperplanes;
- X is isomorphic to its square;
- $X \simeq X \oplus Y$ for every Y generated by a subsequence of the basis.

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(2) X has a minimal subspace, i.e. a subspace Y that can be embedded in every further subspace.

Ferenczi–Rosendal's conjecture

Some progress has been done on this conjecture, even if it is still open:

Definition

A space X is *asymptotically Hilbertian* if there exists $K \geqslant 1$ such that for every $n \in \mathbb{N}$, there exists a finite-codimensional subspace Y of X such that every finite-dimensional subspace E of Y with $\dim(E) \leqslant n$ is K-isomorphic to $\ell_2^{\dim(E)}$.

Theorem (Anisca, 2009)

Every non-Hilbertian, asymptotically Hilbertian space has continuum-many subspaces.

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Theorem (Cuellar, 2016)

Every space with less than continuum-many subspaces is near Hilbert (that is, has type p and cotype q for every $p < 2 < q$).

A dummy reasoning

Questions

(1) *Let X be a space with exactly two subspaces. Does X necessarily have an unconditional basis?*

(2) *Let X be a non-Hilbertian space with less than continuum-many subspaces. Does X necessarily have a non-Hilbertian subspace with an unconditional basis?*

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A positive answer to (2) would also provide a positive answer to (1), since by a result by Anisca (2007), a ℓ_2 can be embedded in every space with finitely many subspaces.

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Since an HI space contains no minimal subspace, it has to contain continuum-many subspaces. Hence, by Gowers' dichotomy, a space with less than continuum-many subspaces must contain a subspace with an unconditional basis...

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Since an HI space contains no minimal subspace, it has to contain continuum-many subspaces. Hence, by Gowers' dichotomy, a space with less than continuum-many subspaces must contain a subspace with an unconditional basis... But this subspace could be Hilbertian. How to avoid this case?

A not so dummy reasonning, finally

The "only" ingredient we need in the proof of Gowers' dichtomy is the fact that, given a space X and a decreasing sequence of subspaces $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$, there exists a subspace X_∞ such that $\forall n \in \mathbb{N}\, X_\infty \subseteq^* X_n$ (where $Y \subseteq^* Z$ means that $Y \cap Z$ has finite codimension in Y).

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But... This also works if we restrict our attention to non-Hilbertian subspaces !

The first dichotomy

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Definition

A space X is Hereditarily Hilbert-primary (HHP) if for every subspaces $Y, Z \subseteq X$, if Y and Z are in topological direct sum, then either Y or Z is Hilbertian.

Theorem

Let X be a non-Hilbertian space. Then there exists a non-Hilbertian subspace Y of X such that:

- *either Y has an good UFDD;*
- *or Y is HHP.*

Moreover, these two cases are mutually exclusive.

What are the consequences for the number of subspaces ?

The first dichotomy

Proposition

Let X be a non-Hilbertian space with less than continuum-many subspaces. Suppose X has a UFDD. Then X has a non-Hilbertian subspace with an unconditional basis.

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Corollary

Let X be a non-Hilbertian space with less than continuum-many subspaces. Then either exists a non-Hilbertian subspace Y of X such that:

- *either Y has an unconditional basis;*
- *or Y is HHP.*

In particular, if we manage to prove that an HHP space must have at least three subspaces, then our first conjecture is true, and if all HHP spaces have continuum-many subspaces, then our second conjecture is true.

HHP spaces

Known proofs that HI spaces are not isomorphic to any of their proper subspaces don't seem to adapt easily to the case of HHP spaces. We are still not able to prove that such spaces have at least three subspaces.

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The following spaces are the only examples of HHP spaces we currently know:

- Trivial ones: ℓ_2, HI spaces.
- $X \oplus \ell_2$, for X an HI space.
- A HI sum of spaces isomorphic to ℓ_2 (Argyros–Raikoftsalis, 2008). The space $\mathfrak{X}_2$ they construct actually admits a unique decomposition as $\mathfrak{X}_2 \oplus \ell_2$.

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All these spaces contain an HI subspace, so they have continuum-many subspaces.

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All these spaces contain an HI subspace, so they have continuum-many subspaces.

Question

Does there exist ℓ_2-saturated HHP spaces that are non-Hilbertian?

The second dichotomy

In order to reduce the question of the number of subspaces of HHP spaces to something simpler, we introduce a second dichotomy. This is a Hilbert-avoiding version of the minimal/tight dichotomy by Ferenczi and Rosendal.

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Definition (Ferenczi – Rosendal)

Let (e_i) be a basis.

- A space Y is tight in (e_i) if there are successive intervals of integers $I_0 < I_1 < \dots$ such that for every infinite $A \subseteq \mathbb{N}$, Y does not embed into $\overline{\operatorname{span}}(e_i \mid i \notin \bigcup_{j \in A} I_j)$.
- The basis (e_i) is tight if every space is tight in it. A space is tight if it has a tight basis.

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- The basis (e_i) is tight if every space is tight in it. A space is tight if it has a tight basis.

Theorem (Ferenczi–Rosendal, 2009)

Every Banach space either contains a minimal subspace, or a tight subspace.

The second dichotomy

Definition

A space X is minimal among non-Hilbertian spaces (MNH) if it is non-Hilbertian and embeds in all of its non-Hilbertian subspaces.

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Let (E_i) be a FDD.

- A space Y is tight in (E_i) if there are successive intervals of integers $I_0 < I_1 < \dots$ such that for every infinite $A \subseteq \mathbb{N}$, Y does not embed into $\overline{\bigoplus_{i \notin \bigcup_{j \in A} I_j} E_i}$.
- The FDD (E_i) is tight for non-Hilbertian spaces (TNH) if every non-Hilbertian is tight in it. A space is tight for non-Hilbertian spaces (TNH) if it has good FDD which is TNH.

The second dichotomy

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Every non-Hilbertian space contains either a MNH subspace, or a TNH subspace.

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Proposition

Every TNH space has continuum-many subspaces.

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Corollary

A space with less than continuum-many subspaces has a MNH subspace.

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Every non-Hilbertian space contains either a MNH subspace, or a TNH subspace.

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Every TNH space has continuum-many subspaces.

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A space with less than continuum-many subspaces has a MNH subspace.

So to prove our second conjecture, it would be enough to show that an HHP space cannot be MNH.

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Every non-Hilbertian space contains either a MNH subspace, or a TNH subspace.

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Every TNH space has continuum-many subspaces.

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A space with less than continuum-many subspaces has a MNH subspace.

So to prove our second conjecture, it would be enough to show that an HHP space cannot be MNH.

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Are there non-trivial (ℓ_2-saturated) MNH spaces?

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1. *Let X be an MNH space. Then there exist a constant C such that X embeds C-isomorphically into every non-Hilbertian subspace of itself.*

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Proposition

1. *Let X be an MNH space. Then there exist a constant C such that X embeds C-isomorphically into every non-Hilbertian subspace of itself.*
2. *An asymptotically Hilbertian space cannot contain an MNH subspace.*

The latter result implies the result by Anisca about asymptotically Hilbertian spaces.

MNH spaces

Proposition

Let X be an MNH space. Then:

1. *X has a non-Hilbertian subspace with a Schauder basis;*
2. *If X has a UFDD, then X has a non-Hilbertian subspace with an unconditional basis.*

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Question

Does these last results remain true if we remove the MNH hypothesis?

Thank you for your attention!

Joyeux anniversaire Robert !