# Global aspects of unitary representations

Michal Doucha

Czech Academy of Sciences
joint work in progress with Maciej Malicki

June 20, 2017

# Introduction-the full generality

Let $\Gamma$ be a countable discrete group and $G$ a topological group. The set of all homomorphisms of $\Gamma$ into $G$ may be identified with a closed subspace, denoted by $\mathrm{Rep}(\Gamma, G)$, of the topological space $G^\Gamma$.

# Introduction-the full generality

Let $\Gamma$ be a countable discrete group and $G$ a topological group. The set of all homomorphisms of $\Gamma$ into $G$ may be identified with a closed subspace, denoted by $\mathrm{Rep}(\Gamma, G)$, of the topological space $G^\Gamma$.

This has been investigated for example when $G = \mathrm{GL}(n, \mathbb{K})$ or $G = U(H)$. More generally, recently it has been considered for $G = \mathrm{Aut}(X)$, where $X$ is some countable structure, e.g. set, graph, etc.

# Introduction-the full generality

Let $\Gamma$ be a countable discrete group and $G$ a topological group. The set of all homomorphisms of $\Gamma$ into $G$ may be identified with a closed subspace, denoted by $\mathrm{Rep}(\Gamma, G)$, of the topological space $G^\Gamma$.

This has been investigated for example when $G = \mathrm{GL}(n, \mathbb{K})$ or $G = U(H)$. More generally, recently it has been considered for $G = \mathrm{Aut}(X)$, where $X$ is some countable structure, e.g. set, graph, etc.

If $G$ is Polish, then $\mathrm{Rep}(\Gamma, G)$ is also Polish, thus a Baire space and one may consider properties of $\mathrm{Rep}(\Gamma, G)$ that are satisfied by meager, resp. comeager many elements.

# Generic homomorphisms

Of particular interest is the question whether there are generic homomomorphisms. Call two homomorphisms $\pi_1, \pi_2 \in \mathrm{Rep}(\Gamma, G)$ *equivalent* if there is $g \in G$ such that

$$\pi_1(x) = g \cdot \pi_2(x) \cdot g^{-1}$$

for all $x \in \Gamma$.

# Generic homomorphisms

Of particular interest is the question whether there are generic homomomorphisms. Call two homomorphisms $\pi_1, \pi_2 \in \mathrm{Rep}(\Gamma, G)$ *equivalent* if there is $g \in G$ such that

$$\pi_1(x) = g \cdot \pi_2(x) \cdot g^{-1}$$

for all $x \in \Gamma$.

We are interested whether there are homomorphisms with comeager equivalence classes, or on the other hand all equivalence classes are meager. In the former case we say that $\Gamma$ has a generic homomorphism/representation.

# Generic homomorphisms

Of particular interest is the question whether there are generic homomomorphisms. Call two homomorphisms $\pi_1, \pi_2 \in \mathrm{Rep}(\Gamma, G)$ *equivalent* if there is $g \in G$ such that

$$\pi_1(x) = g \cdot \pi_2(x) \cdot g^{-1}$$

for all $x \in \Gamma$.

We are interested whether there are homomorphisms with comeager equivalence classes, or on the other hand all equivalence classes are meager. In the former case we say that $\Gamma$ has a generic homomorphism/representation.

**Theorem (Y. Glasner, Kitroser, Melleray, 2016)**

A countable discrete $\Gamma$ has a generic permutation representation (i.e. comeager class in $\mathrm{Rep}(\Gamma, S_{\mathbb{N}})$) iff $\Gamma$ is solitary (LERF implies solitary).

# Generic homomorphisms

**Theorem (Rosendal, 2011)**

If $\Gamma$ has the Ribes-Zalesskij property, then $\Gamma$ has a generic representation in $\mathrm{Rep}(\Gamma, \mathrm{Aut}(\mathbb{QU}))$.

**Theorem (Rosendal, 2011)**

If $\Gamma$ has the Ribes-Zalesskij property, then $\Gamma$ has a generic representation in $\mathrm{Rep}(\Gamma, \mathrm{Aut}(\mathbb{QU}))$.

Here we are interested in the case $G = U(H)$, where $H$ is a separable infinite-dimensional Hilbert space and we write $\mathrm{Rep}(\Gamma, H)$ instead of $\mathrm{Rep}(\Gamma, U(H))$.

**Theorem (Rosendal, 2011)**

If $\Gamma$ has the Ribes-Zalesskij property, then $\Gamma$ has a generic representation in $\mathrm{Rep}(\Gamma, \mathrm{Aut}(\mathbb{QU}))$.

Here we are interested in the case $G = U(H)$, where $H$ is a separable infinite-dimensional Hilbert space and we write $\mathrm{Rep}(\Gamma, H)$ instead of $\mathrm{Rep}(\Gamma, U(H))$.

**Theorem (folklore? Rokhlin?)**

Every conjugacy class in $U(H)$ is meager.

# Generic homomorphisms

Notice that $U(H)$ is naturally homeomorphic with $\mathrm{Rep}(\mathbb{Z}, H)$ (analogously, $U(H)^n$ is naturally homeomorphic with $\mathrm{Rep}(F_n, H)$). So it follows and is known:

**Corollary**

If $\Gamma$ is a finitely generated free group, then all equivalence classes in $\mathrm{Rep}(\Gamma, H)$ are meager.

# Haagerup property

**Definition**

$\Gamma$ has the Haagerup property (or is a-T-menable) if there exists a sequence $(\phi_n)_n$ of normalized positive definite functions on $\Gamma$ such that

- they vanish at infinity, i.e. $(\phi_n)_n \subseteq c_0(\Gamma)$;
- they converge pointwise to the constant function 1.

# Haagerup property

**Definition**

$\Gamma$ has the Haagerup property (or is a-T-menable) if there exists a sequence $(\phi_n)_n$ of normalized positive definite functions on $\Gamma$ such that

- they vanish at infinity, i.e. $(\phi_n)_n \subseteq c_0(\Gamma)$;
- they converge pointwise to the constant function 1.

Equivalently, $\Gamma$ admits a proper action on a Hilbert space by isometries.

# Haagerup property

**Definition**

$\Gamma$ has the Haagerup property (or is a-T-menable) if there exists a sequence $(\phi_n)_n$ of normalized positive definite functions on $\Gamma$ such that

- they vanish at infinity, i.e. $(\phi_n)_n \subseteq c_0(\Gamma)$;
- they converge pointwise to the constant function 1.

Equivalently, $\Gamma$ admits a proper action on a Hilbert space by isometries.

**Theorem, ??**

$\Gamma$ has the Haagerup property iff the set of mixing representations of $\Gamma$ is dense in $\mathrm{Rep}(\Gamma, H)$.

# Haagerup property

**Theorem**

If an infinite group $\Gamma$ has the Haagerup property and the set of finite-dimensional representations is dense in $\mathrm{Rep}(\Gamma, H)$, then all equivalence classes in $\mathrm{Rep}(\Gamma, H)$ are meager.

# Haagerup property

**Theorem**

If an infinite group $\Gamma$ has the Haagerup property and the set of finite-dimensional representations is dense in $\mathrm{Rep}(\Gamma, H)$, then all equivalence classes in $\mathrm{Rep}(\Gamma, H)$ are meager.

**Idea of the proof.**

For a fixed countable dense subset $D$ of the unit sphere in $H$ and for any normalized positive definite function $\phi$ on $\Gamma$, the set $I_\phi = \{\pi \in \mathrm{Rep}(\Gamma, H) : \forall \xi \in D\ \exists^\infty x \in \Gamma\ (|\phi(x) - \phi_{\pi,\xi}(x)| > 1/4\}$ is dense $G_\delta$.

# Representations of C*-algebras

Let $A$ be a separable (unital) C*-algebra. Then the set of all representations of $A$ in $B(H)$ is a Polish space (a Polish subset of the non-metrizable space $B(H)^A$).

## Facts

- For a countable discrete group $\Gamma$ the spaces $\mathrm{Rep}(\Gamma, H)$ and $\mathrm{Rep}(C^*(\Gamma), H)$ are naturally homeomorphic.
- (Exel, Loring) A C*-algebra $A$ is residually finite-dimensional iff finite-dimensional representations are dense in $\mathrm{Rep}(A, H)$.

# Representations of C*-algebras

Let $A$ be a separable (unital) C*-algebra. Then the set of all representations of $A$ in $B(H)$ is a Polish space (a Polish subset of the non-metrizable space $B(H)^A$).

## Facts

- For a countable discrete group $\Gamma$ the spaces $\mathrm{Rep}(\Gamma, H)$ and $\mathrm{Rep}(C^*(\Gamma), H)$ are naturally homeomorphic.
- (Exel, Loring) A C*-algebra $A$ is residually finite-dimensional iff finite-dimensional representations are dense in $\mathrm{Rep}(A, H)$.

## Restatement of the theorem

Let $\Gamma$ be a countably infinite discrete group with the Haagerup property and suppose that the full group C*-algebra $C^*(\Gamma)$ is residually finite-dimensional. Then all equivalence classes in $\mathrm{Rep}(C^*(\Gamma), H)$ are meager.

# Representations of C*-algebras

## Theorem

Let $A$ be a separable infinite-dimensional unital C*-algebra which is residually finite-dimensional and such that its unitary group contains a countable dense subgroup with the Haagerup property. Then all equivalence classes in $\mathrm{Rep}(A, H)$ are meager.

## Corollary

If $A$ is an abelian infinite-dimensional unital C*-algebra, then all equivalence classes in $\mathrm{Rep}(A, H)$ are meager.

### Theorem

Let $A$ be a separable infinite-dimensional unital C*-algebra which is residually finite-dimensional and such that its unitary group contains a countable dense subgroup with the Haagerup property. Then all equivalence classes in $\mathrm{Rep}(A, H)$ are meager.

### Corollary

If $A$ is an abelian infinite-dimensional unital C*-algebra, then all equivalence classes in $\mathrm{Rep}(A, H)$ are meager.

### Theorem

If $(A, \alpha, \Gamma)$ is a C*-dynamical system, where $\Gamma$ is a countably infinite discrete group with the Haagerup property and $B = C^*(A, \alpha, \Gamma)$ is residually finite-dimensional. Then all equivalence classes in $\mathrm{Rep}(B, H)$ are meager.

# Kazhdan's property

**Definition**

A countable discrete group $\Gamma$ has the Kazhdan's property T if there are a finite set $F \subseteq \Gamma$ and $\varepsilon > 0$ such that whenever $\pi \in \mathrm{Rep}(\Gamma, H)$ has an $(F, \varepsilon)$-almost invariant unit vector, then it has an invariant vector.
Equivalently, if $1_\Gamma \preceq \pi$, then $1_\Gamma \leq \pi$.
Kerr-Pichot: $\Gamma$ does not have the Kazhdan's property iff weakly mixing representations form a dense $G_\delta$ subset of $\mathrm{Rep}(\Gamma, H)$.

# Kazhdan's property

## Definition

A countable discrete group $\Gamma$ has the Kazhdan's property T if there are a finite set $F \subseteq \Gamma$ and $\varepsilon > 0$ such that whenever $\pi \in \mathrm{Rep}(\Gamma, H)$ has an $(F, \varepsilon)$-almost invariant unit vector, then it has an invariant vector.
Equivalently, if $1_\Gamma \preceq \pi$, then $1_\Gamma \leq \pi$.
Kerr-Pichot: $\Gamma$ does not have the Kazhdan's property iff weakly mixing representations form a dense $G_\delta$ subset of $\mathrm{Rep}(\Gamma, H)$.

## Fact

Invariant vectors are 'close' to the almost invariant ones. That is, if $\xi \in H$ is a unit $(F, \delta \cdot \varepsilon)$-almost invariant vector for $\pi \in \mathrm{Rep}(\Gamma, H)$, then there is an invariant vector $\xi' \in H$ such that $\|\xi - \xi'\| < \delta$.

# Kazhdan's property

**Theorem (Wang)**

If $\Gamma$ has the Kazhdan's property and $\sigma$ is a finite-dimensional irreducible unitary representation of $\Gamma$, then for any $\pi \in \mathrm{Rep}(\Gamma, H)$ we have that if $\sigma \preceq \pi$, then $\sigma \leq \pi$.

**Theorem (Wang)**

If Γ has the Kazhdan's property and $\sigma$ is a finite-dimensional irreducible unitary representation of Γ, then for any $\pi \in \mathrm{Rep}(\Gamma, H)$ we have that if $\sigma \preceq \pi$, then $\sigma \leq \pi$.

**Question**

Let $\phi$ be a positive definite function on Γ associated to some finite-dimensional irreducible representation $\sigma$ of Γ. Let $\pi \in \mathrm{Rep}(\Gamma, H)$ and $\xi \in H$, $\|\xi\| = 1$, be such that $\phi$ and $\phi_{\pi,\xi}$ are 'very close' on some finite subset of Γ. Does there exist $\xi' \in H$ 'close' to $\xi$ such that the subrepresentation of $\pi$ induced by $\xi'$ is equivalent to $\sigma$?

# Kazhdan's property

**Theorem**

Let $\Gamma$ be a countable discrete Kazhdan group with densely many finite-dimensional representations ($C^*(\Gamma)$ is residually finite-dimensional). Suppose that the answer to the question is positive. Then $\Gamma$ has a generic representation, i.e. a representation with comeager equivalence class. It is the direct sum of all finite-dimensional irreducible representations, each with infinite multiplicity.

# Kazhdan's property

**Theorem**

Let Γ be a countable discrete Kazhdan group with densely many finite-dimensional representations ($C^*(\Gamma)$ is residually finite-dimensional). Suppose that the answer to the question is positive. Then Γ has a generic representation, i.e. a representation with comeager equivalence class. It is the direct sum of all finite-dimensional irreducible representations, each with infinite multiplicity.

**Theorem**

Let Γ be a countable discrete group. Suppose that some $\pi \in \mathrm{Rep}(\Gamma, H)$ has a comeager equivalence class. Then $\pi[\Gamma]$ is a discrete subgroup of $U(H)$.

The generic representation $\pi_G$ factorizes through the direct product of finite-dimensional unitary groups, thus its image in $U(H)$ is pre-compact. However, it is discrete, thus it is finite. A contradiction.

The generic representation $\pi_G$ factorizes through the direct product of finite-dimensional unitary groups, thus its image in $U(H)$ is pre-compact. However, it is discrete, thus it is finite. A contradiction.

Recall

**Question (Lubotzky, Shalom)**

Does there exist a Kazhdan group with property FD?

Property FD of $\Gamma$ is stronger than $C^*(\Gamma)$ being residually finite-dimensional.