Differentiability of typical Lipschitz functions.

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  $\mathrm{Lip}_1([0,1]^d)$ becomes a complete metric space.
- A *typical* Lipschitz function $f \in \mathrm{Lip}_1([0,1]^d)$ is an element of a residual subset of $\mathrm{Lip}_1([0,1]^d)$.

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Theorem (Rademacher's Theorem)

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function. Then f is differentiable everywhere except for a set of Lebesgue measure zero.

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Every set in $\mathbb{R}^d$ of positive Lebesgue measure captures points of differentiability of every Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$.

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1. *A cannot be covered by a countable union of closed sets of Lebesgue measure zero.*
2. *The typical Lipschitz function $f \in \mathrm{Lip}_1([0, 1])$ has points of differentiability in A.*

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Purely unrectifiable sets.

Definition

A set $P \subseteq \mathbb{R}^d$ is called *purely unrectifiable* (p.u.) if for every Lipschitz curve $\gamma\colon [0,1] \to \mathbb{R}^d$ the set $\gamma^{-1}(P)$ has Lebesgue measure zero.

Extreme non-differentiability in p.u. sets.

Theorem (Alberti, Csörnyei, Preiss, 2010)

Let $P \subseteq \mathbb{R}^d$ be a Borel, p.u. set. Then there is a Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$ such that f has no directional derivatives in P.

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Theorem (Preiss, Maleva, 2017)

Let P be an F_σ, p.u. set. Then there is a Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$ which is non-differentiable in the strongest possible sense at every point of P.

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1. *the typical* $f \in \mathrm{Lip}_1([0,1]^d)$ *has differentiability points in* A*, or*
2. *the typical* $f \in \mathrm{Lip}_1([0,1]^d)$ *is nowhere differentiable in* A*.*

Comparison with universal differentiability sets.

Definition

A set $U \subseteq \mathbb{R}^d$ is called a *universal differentiability set* (UDS) if U contains a point of differentiability of every Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$.

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- By Rademacher's theorem, every set of positive measure is a UDS.
- If $d > 1$ then there exist UDS in $\mathbb{R}^d$ with Lebesgue measure zero. (Preiss, 1990)

Many points of differentiability in a UDS.

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Let $U \subseteq \mathbb{R}^d$ be a UDS and $f\colon \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function. Then

- $\mathrm{Diff}(f) \cap U$ has one-dimensional projections of positive measure in every direction.
- $\Rightarrow \mathrm{Diff}(f) \cap U$ has positive one-dimensional Hausdorff measure.

P.u. sets cannot capture too many differentiability points.

Theorem (D., Maleva, 2019)

Let $U \subseteq (0,1)^d$ *be a UDS and* $V \subseteq U$ *be a set for which*

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$$\mathrm{Diff}(f) \cap U \subseteq V,$$

for the typical $f \in \mathrm{Lip}_1([0,1]^d)$. *Then* V *is a UDS*$\Rightarrow$ V *is not p.u.*

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1. $\mathrm{Diff}(f) \cap P$ *projects in every direction to a set of positive measure.*
2. $\mathrm{Diff}(f) \cap P$ *has non-*σ*-finite one dimensional Hausdorff measure.*

Thank you for you attention!