Mapping n grid points onto a square forces an arbitrarily large Lipschitz constant.

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Universität Innsbruck

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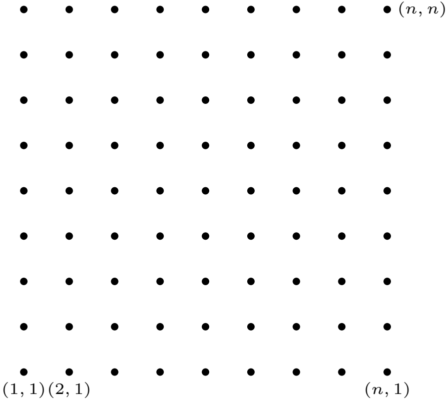
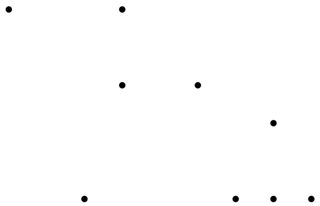
Joint work with Vojtěch Kaluža and Eva Kopecká.

‘The regular $n \times n$ grid’ Q_n.



Best possible Lipschitz constants.



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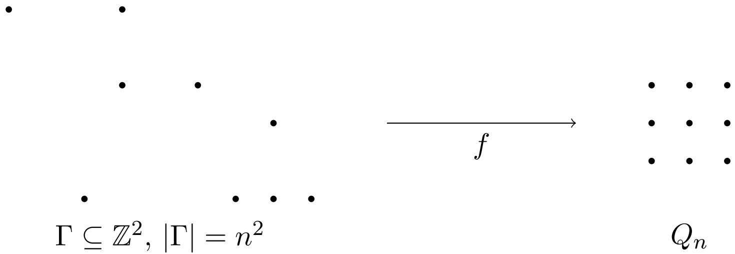

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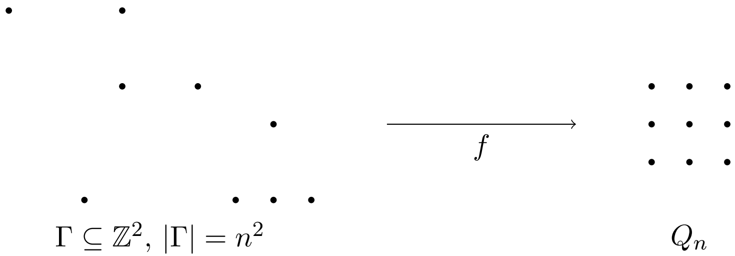

Problem Determine

(i) $L_{n,\Gamma} := \min \{\mathrm{Lip}(f) \colon f : \Gamma \to Q_n \text{ bijective}\}$,

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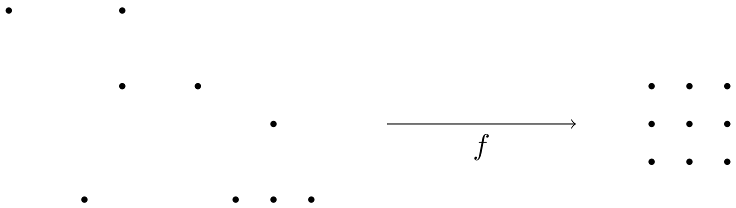

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Problem Determine

(i) $L_{n,\Gamma} := \min \{\mathrm{Lip}(f) \colon f \colon \Gamma \to Q_n \text{ bijective}\}$,

(ii) $L_n := \sup_{\Gamma} L_{n,\Gamma}$.

Feige's Question.

Is the sequence $(L_n)_{n=1}^{\infty}$ bounded?

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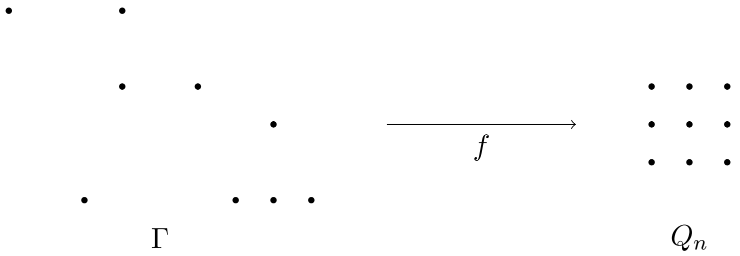
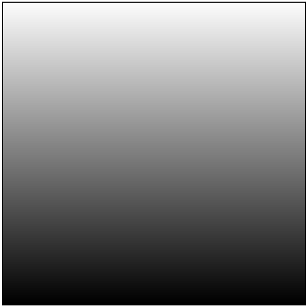


Does there exist $L > 0$ such that for any $n \in \mathbb{N}$ and any set $\Gamma \subseteq \mathbb{Z}^2$ with $|\Gamma| = n^2$ there exists a bijective mapping $f\colon \Gamma \to Q_n$ with $\mathrm{Lip}(f) \leq L$?

Densities as separated sets.



density $\rho\colon [0,1]^2 \to (0,\infty)$

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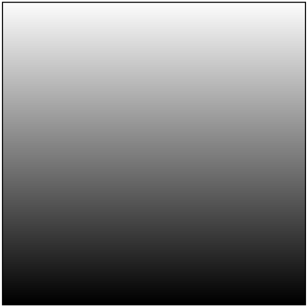

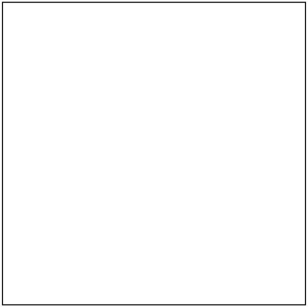

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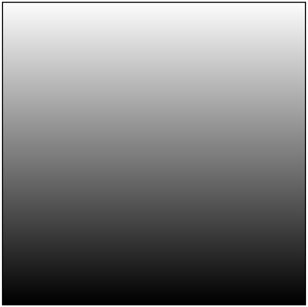

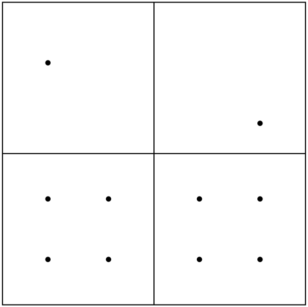

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S_1

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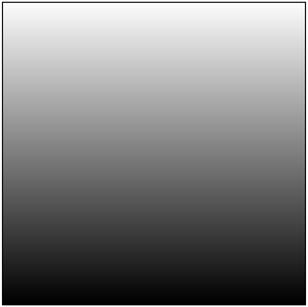

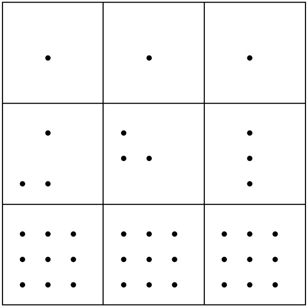

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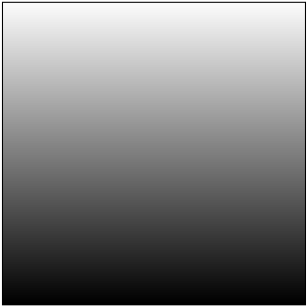

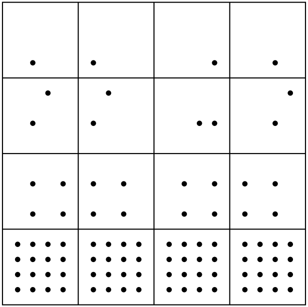

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S_k

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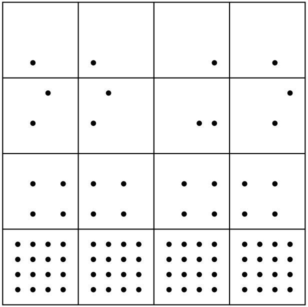

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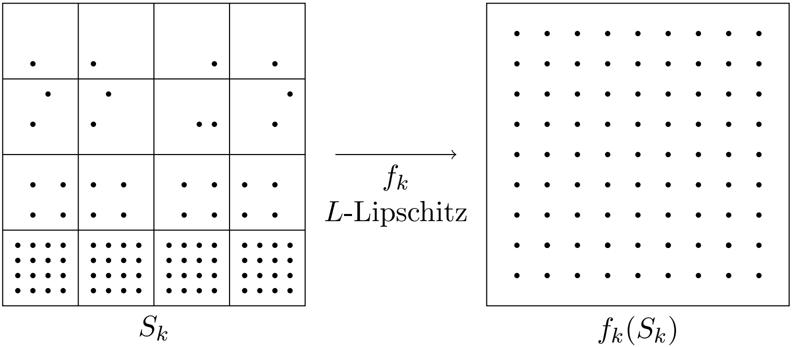

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Lemma (D., Kaluža, Kopecká (2017))

Suppose that the answer to Feige's question is positive. Then for every measurable density $\rho\colon [0,1]^2 \to (0,\infty)$ *with* $0 < \inf \rho < \sup \rho < \infty$ *there exists a Lipschitz regular mapping* $f\colon [0,1]^2 \to \mathbb{R}^2$ *such that*

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Non-bilipschitz equivalent separated nets.

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$$\rho = |\mathrm{Jac}(f)| \qquad a.e.$$

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Lipschitz Regular Mappings.

- We call a mapping $f\colon A \subseteq \mathbb{R}^d \to \mathbb{R}^d$ *Lipschitz regular* if f is Lipschitz and there exists a constant $C > 0$ such that the preimage $f^{-1}(B)$ of any ball $B \subseteq \mathbb{R}^d$ can be covered by C balls of the same radius.

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- We will call a Lipschitz mapping (C, L)-Lipschitz regular if it is L-Lipschitz and regular with constant C.

Bilipschitz behaviour of regular mappings.

Let $B \subseteq \mathbb{R}^d$ be a non-empty open ball and $f\colon B \to \mathbb{R}^d$ be a (C, L)-Lipschitz regular mapping.

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Theorem (D., Kaluža, Kopecká (2017))

There exists a non-empty, open ball $B' \subseteq B$ such that $f|_{B'}$ is bilipschitz with lower bilipschitz constant $b = b(C)$.

Porous and σ-porous sets.

Definition

Let (M, d) be a complete metric space.

(i) A set $E \subseteq M$ is called *porous* if there exists $c \in (0, 1)$ such that for every $\varepsilon > 0$ and every $x \in E$ there exists $y \in M$ with $d(x, y) < \varepsilon$ and $B(y, c\varepsilon) \cap E = \emptyset$.

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(ii) A set $F \subseteq M$ is called σ-*porous* if F can be written as a countable union of porous sets.

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Theorem (D., Kaluža, Kopecká (2017))

$\mathcal{E}$ is a σ-porous subset of $C([0,1]^2)$.

Thank you for your attention!