On some extensions to Banach spaces of the notion of Hilbert-Schmidt operator

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Métabief - 20 juin 2019

Dedicated to Robert Deville for his 60th birthday

Definition: *Let* H_1 *et* H_2 *be Hilbert spaces. An operator* $T : H_1 \to H_2$ *is a Hilbert-Schmidt operator if, for at least an orthonormal basis* $(e_i)_{i\in I}$ *of* H_1

$$\sum_{i\in I} \|Te_i\|^2 < +\infty,$$

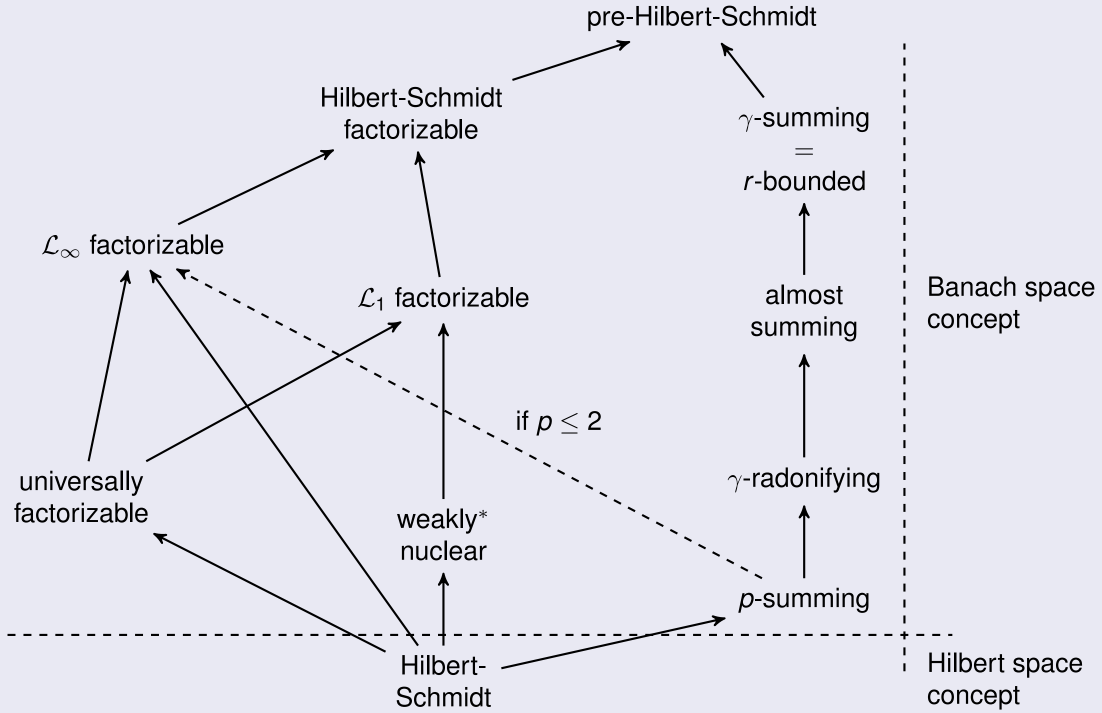
and in this case this condition is in fact satisfied by all orthonormal bases of H_1.

The space of Hilbert-Schmidt operators from H_1 into H_2 is denoted $S_2(H_1, H_2)$.

Definition: *Let* E *and* F *be Banach spaces. An operator* $u : E \to F$ *is a pre-Hilbert-Schmidt operator if* $w \circ u \circ v$ *is Hilbert-Schmidt whenever* H_1, H_2 *are Hilbert spaces and* $v : H_1 \to E$ *and* $w : F \to H_2$ *are bounded linear operators.*

The space of pre-Hilbert-Schmidt operators from E into F is denoted $PS_2(E, F)$. If E and F are Hilbert spaces, then $PS_2(E, F) = S_2(E, F)$.

The general picture



These implications become equalities in the Hilbert space context. On the other hand the fact that these various conditions on a bounded linear operator imply that the operator is pre-Hilbert-Schmidt follow from the fact that the classes under consideration are stable under left and right composition with bounded operators.

p-summing operators

Definition: *Let* E, F *be Banach spaces, and let* $p \in [1, +\infty)$. *An operator* $u : E \to F$ *is said to be p-summing if and only if there exists* $c \geq 0$ *such that for* $m \geq 1$, $x_1, \ldots, x_m \in X$, *we have:*

$$\sum_{i=1}^{m} \|ux_i\|^p \leq c^p \sup_{x^* \in X^*, \|x^*\| \leq 1} \sum_{i=1}^{m} | < x_i, x^* > |^p. \quad (1)$$

Hence the class $\Pi_p(E, F)$ of p-summing operators from E into F is the class of operators which turn "weakly-p-summing sequences" in E into p-summing sequences in F. The smallest constant $c \geq 0$ satisfying (1) is denoted $\pi_p(u)$. It follows from the Pietsch factorization theorem that we have the following characterization of p-summing operators.

Theorem: *A bounded operator* $u : E \to F$ *is p-summing if and only if there exists a weakly closed (hence compact) norming subset of the unit ball of* E^*, *a probability measure* μ *on* K *and a bounded operator* $\tilde{u} : L^p(\mu) \to Y$ *such that* $u \circ j_p \circ i_E = u$, *where* $i_E : E \to \mathcal{C}(K)$ *and* $j_p : \mathcal{C}(K) \to L^p(\mu)$ *denote the natural injections, and in this case* $\pi_p(u) = \|\tilde{u}\|$.

absolutely summing and γ-summing operators

A vector-valued series $\sum_{n=1}^{+\infty} x_n$ is said to be commutatively convergent if the series $\sum_{n=1}^{+\infty} x_{\sigma_n}$ is convergent for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$, and an operator $u : E \to F$ is absolutely summing if $\sum_{n=1}^{+\infty} \|x_n\| < +\infty$ for every commutatively convergent series $\sum_{n=1}^{+\infty} x_n$.The space of $\Pi_{abs}(E, F)$ of absolutely summing operators from E into F has a natural norm $\|.\|_{abs}$, and it is a standard fact that $\Pi_{abs}(E, F) = \Pi_1(E, F)$ and that $\|u\|_{abs} = \pi_1(u)$ for every $u \in \Pi_{abs}(E, F)$.
Definition: *An operator* $u : E \to F$ *is said to be* γ*-summing if and only if there exists* $c \geq 0$ *such that for* $m \geq 1, x_1, \ldots, x_m \in X$, *we have:*

$$E\left(\left\|\sum_{i=1}^{m} \gamma_i x_i\right\|^2\right) \leq c^2 \sup_{x^* \in X^*, \|x^*\| \leq 1} \sum_{i=1}^{m} | < x_i, x^* > |^2, \tag{2}$$

where $(\gamma_n)_{n \geq 1}$ *denotes a sequence of independent gaussian variables.*
The space of γ-summing operators from E into F is denoted $\gamma^\infty(E, F)$, and for $u \in \gamma^\infty(E, F)$, $\|u\|_{\gamma^\infty(E,F)}$ is the smallest constant $c \geq 0$ satisfying (2).

Rademacher bounded operators

Definition: *An operator $u : E \to F$ is said to be r-bounded (Rademacher-bounded) if and only if there exists $c \geq 0$ such that for $m \geq 1$, $x_1, \dots, x_m \in X$, we have:*

$$E\left(\left\|\sum_{i=1}^{m} r_i x_i\right\|^2\right) \leq c^2 \sup_{x^* \in X^*, \|x^*\| \leq 1} \sum_{i=1}^{m} | < x_i, x^* > |^2,$$

where $(r_n)_{n\geq 1}$ denotes a Rademacher sequence, i.e. a sequence of independent Random variables taking the values 1 and -1 with probability $1/2$.

The space of r-bounded operators from E into F is denoted $rb(E, F)$, and for $u \in rb(E, F)$, $\|u\|_{rb(E,F)}$ denotes the smallest constant $c \geq 0$ satisfying the above inequality.

Theorem: *We have $\gamma^\infty(E, F) = rb(E, F)$ and*

$$m_1 \|u\|_{rb(E,F)} \leq \|u\|_{\gamma^\infty(E,F)} \|u\|_{rb(E,F)} \quad (u \in rb(E, F)),$$

where $m_1 = \sqrt{\frac{2}{\pi}}$.

almost summing operators and γ-radonifying operators

Let $p \geq 1$, and let E be a Banach space. The weak space $\ell^p_{weak}(E)$ consists in those sequences $(x_n)_{n\geq 1}$ such that

$$\|(x_n)_{n\geq 1}\|_{\ell^p_{weak}(E)} = \sup_{x^* \in X^*, \|x^*\| \leq 1} \left(\sum_{n=1}^{+\infty} | < x_n, x^* > | < +\infty \right)^{1/p}.$$

Definition: *An operator $u \in \gamma^\infty(E, F)(resp.rb(E, F))$ is said to be γ-radonifying (resp. almost summing) if the series $\sum_{n=1}^{+\infty} \gamma_n x_n$ (resp. $\sum_{n=1}^{+\infty} \gamma_n x_n$) converges in $L^2(\Omega, F)$ for every sequence $(x_n)_{n\geq 1} \in \ell^2_{weak}(E)$.*

It follows from a theorem of Itô-Nisio [**?**], th. 2.17 that convergence in $L^2(\Omega, F)$ of these series is equivalent to almost sure convergence, or convergence in $L^p(\Omega, F)$ for any $p \geq 1$. We will denote $\gamma(E, F)$ (resp. $\Pi_{as}(E, F)$ the space of γ-radonifying (resp.almost summing) operators from E into F.

If the space F does not contain any closed subspace isomorphic to c_0, it follows from results of A Hoffman-Jorgensen [**?**] that $\gamma^\infty(E, F) = rb(E, F) = \Pi_a s(E, F) = \gamma(E, F)$. In the other direction it was shown by Linde and Pietsch [**?**] that $\gamma(\ell^2, c_0)$ is strictly included in $\gamma(\ell 2, c_0)$.

Right chain of implications in the diagram

Unfortunately, Diestel, Jarchow and Tonge wrongly assumed in their excellent monograph [**?**] that $rb(E,F) = \Pi_{as}(E,F)$, which is indeed true if F does not have any closed subspace isomorphic to c_0 but is not true in general (this mistake was already pointed out by Blasco, Tarieladze and Vidal in [**?**]).
We have $\Pi_p(E,F) \subset \Pi_q(E,F)$ for $1 \le p \le q < +\infty$, and $\gamma(E,F) \subset \Pi_{as}(E,F)$, Using the Kahane-Khintchine inequalities, Linde and Pietsch showed that $\Pi_p(E,F) \subset \gamma(E,F)$ for $p \ge 1$, and gave a constant c_p satisfying $\|u\|_{\gamma(E,F)} \le c_p \Pi_p(u)$ for $u \in \Pi_p(E,F)$, see Prop. 12.1 in [**?**].
The right chain of implication in the diagram given at the beginning of the talk follows then from the classical fact that if H_1 and H_2 are Hilbert spaces then

$$\gamma(H_1,H_2) = \gamma^\infty(H_1,H_2) = S_2(H_1,H_2) = \Pi_p(H_1,H_2)$$

for $1 \le p \le +\infty$.

Hilbert-Schmidt operators and nuclearity

An operator $u : E \to F$ is said to be *weak**1-nuclear if $u = \sum_{n=1}^{+\infty} x_n^* \otimes x_n$ where $(x_n)_{n\geq 1}$ is a bounded sequence of elements of F and $(x_n^*)_{n\geq 1} \in \ell^1_{weak^*}$ is a sequence of elements of X^*. We then have $ux = \sum_{n=1}^{+\infty} < x, x_n^* > x_n$ for $x \in E$, which is well-defined since $\sum_{n=1}^{\infty} | < x, x_n^* > | < +\infty$. We have the following result

Proposition: *Let H_1 and H_2 be Hilbert spaces. Then a bounded linear operator $u : H_1 \to H_2$ is Hilbert-Schmidt if and only if u is weak*1-nuclear.*

Since every Hilbert-Schmidt operator u can be written under the form $u = \sum \lambda_n e_n^* \otimes e_n$, where $(e_n^*)_{n\geq 1}$ is an orthonormal sequence of elements of H_1, $(e_n)_{n\geq 1}$ is an orthonormal sequence of elements of H_2 and $(\lambda_n)_{n\geq 1} \in \ell^2$, every Hilbert-Schmidt operator is *weak**1-nuclear.

The converse follows from the fact that every *weak**1-nuclear operator factors through ℓ^1, see the details in [**?**], Prop. 3.17, and it follows from Grothendieck's inequality that ℓ^1 is a Hilbert-Schmidt space, see below. This gives the central column of implications in the diagram given at the beginning of the paper.

Hilbert-Schmidt spaces

Definition: *A Banach space X is called a Hilbert-Schmidt space if given two Hilbert spaces H_1 and H_2 every linear operator $u : H_1 \to H_2$ which factors through X is Hilbert-Schmidt.*

Let $p \in [1, +\infty]$, and let $\lambda > 1$. An ifinite dimensional space X is said to be a $\mathcal{L}^p_\lambda$-space if for every finite dimensional subspace E of X there exists a finite dimensional subspace F of X containing E and an isomorphism $\theta : \ell^p(\{1, \dots, dim(F)\}) \to F$ such that $\|\theta\|\|\theta^{-1}\| \leq \lambda$.

It follows from Grothendieck's inequatily that every bounded linear operator $u : \ell^1 \to \ell^2$ is abolutely summing. More generally we have the following classical result, proved for example in Chap. 3 of [**?**] (we denote by K_G the Grothendieck constant).

Theorem:*(i) Every bounded linear operator u from a $\mathcal{L}^1_\lambda$-space E into a $\mathcal{L}^2_\mu$-space F is absolutely summing, and $\pi_1(u) \leq K_G \lambda \mu \|u\|$.*
(ii) If $1 \leq p \leq 2$, then every bounded linear operator u from a $\mathcal{L}^\infty_\lambda$-space E into a $\mathcal{L}^p_\mu$-space F is 2-summing, and $\pi_2(u) \leq K_G \lambda \mu \|u\|$.

This shows that $\mathcal{L}^1_\lambda$ and $\mathcal{L}^\infty_\lambda$-spaces are Hilbert-Schmidt spaces.

A conjecture

Since Hilbert-Schmidt operators are "universally factorizable", see Th. 19.2 in [**?**], this completes the discussion of the implications of the the diagram given at the beginning of the talk.

It is clear that E is Hilbert-Schmidt if and only if E^* is. The class of Hilbert Schmidt spaces contains the classes $\mathcal{L}^1 := \cup_{\lambda>1}\mathcal{L}^1_\lambda$ and $\mathcal{L}^\infty := \cup_{\lambda>1}\mathcal{L}^\infty_\lambda$, but it is much larger. For example if E is a closed subspace of $\mathcal{C}(K)$ such that $\mathcal{C}(K)/E$ is reflexive, then it follows from Th. 15.13 in [**?**] that every bounded linear operator from E into a Banach space of cotype 2 is 2-summing, and so E is a Hilbert-Schmidt space. Similarly if Z is a reflexive subspace of $L^1(\mu)$ then every bounded linear operator from $L^1(\mu)/Z$ into ℓ^2 is absolutely summing, and so $L^1(\mu)/Z$ is Hilbert-Schmidt. So the Banach algebra $H^\infty(\mathbb{D})$ and the disc algebra $\mathcal{A}(\mathbb{D})$ are Hilbert-Schmidt Banach spaces.
On the other hand a Hilbert-Schmidt space cannot be K-convex, see [**?**], p. 443, and so Hilbert-Schmidt spaces have trivial type, by [**?**].

Conjecture: *Every pre-Hilbert-Schmidt operator factors through a Hilbert-Schmidt space*

A characterization of some diagonal pre Hilbert-Schmidt operators from ℓ^p into ℓ^q.

Results of Linde and Pietsch and computations bu Maurey mentioned in [**?**] give a description of γ-summing diagonal operators $u_\sigma : (x_n)_{n\geq 1} \to (\sigma_n x_n)_{n\geq 1}$ associated to a sequence $\sigma = (\sigma_n)_{n\geq 1}$. Using these results we can characterize the sequences σ such that $u_\sigma \in PS_2(\ell^p, \ell^q)$ for $1 \leq p < 2$, $1 \leq q < +\infty$ and for $2 \leq p < +\infty, 2 \leq q < +\infty$ (we refer to [**?**] for the case where $p = +\infty$ and/or $q = +\infty$). The details are given in [**?**].

p	q	$u_\sigma \in PS_2(\ell^p, \ell^q)$
$1 \leq p < 2$	$1 \leq q < \frac{2p}{2-p}$	$\sigma \in \ell_r, \frac{1}{r} = \frac{1}{2} - \frac{1}{p} + \frac{1}{q}$
$1 \leq p < 2$	$q \geq \frac{2p}{2-p}$	$\sigma \in \ell_\infty$
$2 \leq p < +\infty$	$q \geq 2$	$\sigma \in \ell_q$

THANK YOU FOR YOUR ATTENTION

URTEBETETZE ON!

- S.A. Abdillah, J. Esterle, B. Haak, *Sur quelques extensions au cadre Banachique de la notion d'opérateur de Hilbert-Schmidt*, Studia Math. 227 (2015),192-218.
- O. Blasco, V. Tarieladze, R. Vidal, *K-convexity and duality for almost summing operators* Georgian Math. J. 7(2000), 245-268.
- J. Diestel, H. Jarchow, A. Tonge, Absolutely summing operators. Cambridge Stud ies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995.
- J. Hoffmann-Jorgensen,*J. Sums of independent Banach space valued random variables*, Studia Math. 52(1974)159 - 186.
- W. Linde, A. Pietsch,*Mappings of Gaussian measures of cylindrical sets in Ba- nach Spaces,* Teor.Verojatnost.i Primenen.19 (1974), 472-487, English translation in Theory Probab.Appl.19(1974), 445-460.
- G. Pisier, *Holomorphic semigroups and the geometry of Banach spaces*, Ann Math. 115 (1982), 375-392.
- J. Van Neerven, Radonifying Operators, a survey, The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, 1-61, Proc. Centre Math. Appl. Austral. Nat. Univ., 44, Austral. Nat. Univ., Canberra, 2010.