A characterisation of the Daugavet property in spaces of Lipschitz functions

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- Examples of spaces with the Daugavet property: $C[0,1]$ (Daugavet, 1963), $L_1[0,1]$ (Lozanovskii, 1966), $L_\infty[0,1]$ (Pelczynski, 1965).
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- X has the Daugavet property whenever X^* has the Daugavet property.

Theorem (Kadets-Shvidkoy-Sirotkin-Werner, 2000)

X has the Daugavet property if and only if for every $x_0 \in S_X$, every $\varepsilon > 0$ and every slice S of B_X there exists $x \in S$ such that $||x_0 + x|| > 2 - \varepsilon$.

Spaces of Lipschitz functions

- Given a metric space (M, d) and a distinguished point $0 \in M$, the space

$$\mathrm{Lip}_0(M) := \{f : M \to \mathbb{R} : f \text{ is Lipschitz}, f(0) = 0\}$$

is a dual Banach space when equipped with the norm

$$||f||_L := \sup\left\{\frac{||f(x) - f(y)||}{d(x, y)} : x \neq y\right\}.$$

- The canonical predual of $\mathrm{Lip}_0(M)$ is the Lipschitz-free space $\mathcal{F}(M) = \overline{\mathrm{span}}\{\delta_x : x \in M\}$.
- We will assume that M is **complete**.

Daugavet property for spaces of Lipschitz functions

Note that $\mathrm{Lip}_0([0,1])$ is isometric to $L_\infty[0,1]$ and thus it has the Daugavet property.

Does $\mathrm{Lip}_0([0,1]^2)$ have the Daugavet property? (D. Werner, 2001)

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This problem was solved by Ivakhno, Kadets and Werner in 2006.

A metric space M is said to be

- a *length space* if for every $x, y \in M$, the distance $d(x, y)$ is equal to the infimum of the length of rectifiable curves joining them. Moreover, if that infimum is always attained then we will say that M is a *geodesic space*.
- *local* if for every $\varepsilon > 0$ and every $f \in \mathrm{Lip}_0(M)$ there exist $u, v \in M$ such that $0 < d(u, v) < \varepsilon$ and $\frac{f(u)-f(v)}{d(u,v)} > \|f\|_L - \varepsilon$.
- *spreadingly local* if for every $\varepsilon > 0$ and every $f \in \mathrm{Lip}_0(M)$ the set
$$\left\{ x \in M : \inf_{\delta > 0} \left\| f|_{B(x,\delta)} \right\|_L > \|f\|_L - \varepsilon \right\}$$
is infinite.
- *(Z)* if for every $\varepsilon > 0$ and every $x, y \in M$, $x \neq y$, there is $z \in M \setminus \{x, y\}$ such that
$$d(x, z) + d(z, y) \leq d(x, y) + \varepsilon \min\{d(x, z), d(z, y)\}$$

Theorem (Ivakhno–Kadets–Werner, 2006)

See diagram.

Main result

Theorem (G.-L. – Procházka – Rueda-Zoca)

Let M be a complete metric space. The following are equivalent:

(i) *M is a length space.*

(ii) $\mathrm{Lip}_0(M)$ *has the Daugavet property.*

(iii) $\mathcal{F}(M)$ *has the Daugavet property.*

Remark that a complete metric M is a length space if, and only if, the following condition hold:

$$\forall x, y \in M \; \forall \delta > 0 \; \exists z \in M : \max\{d(x,z), d(y,z)\} \leq \frac{1+\delta}{2} d(x,y)$$

Assume that $\mathcal{F}(M)$ has the Daugavet property. Then M is local.

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Step 1. For every $f \in S_{\mathrm{Lip}_0(M)}$, $\varepsilon > 0$ and $x, y \in M$ there are $u, v \in M$ such that $\frac{f(u)-f(v)}{d(u,v)} > 1 - \varepsilon$ and

$$(1-\varepsilon)(d(x,y)+d(u,v)) < \min\{d(x,v)+d(u,y), d(x,u)+d(v,y)\}$$

Sketch of the proof: The geometric characterisation of the Daugavet property provides $u, v \in M$ such that $\frac{\delta_u - \delta_v}{d(u,v)} \in S(B_{\mathcal{F}(M)}, f, \varepsilon)$ and

$$\left\| \frac{\delta_x - \delta_y}{d(x,y)} + \frac{\delta_u - \delta_v}{d(u,v)} \right\| > 2 - \varepsilon$$

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$$(1-\varepsilon)(d(x,y) + d(u,v)) < \min\{d(x,v) + d(u,y), d(x,u) + d(v,y)\}$$

Step 2. For every $f \in S_{\mathrm{Lip}_0(M)}$, $\varepsilon > 0$ and $x, y \in M$ such that $\frac{f(x)-f(y)}{d(x,y)} > 1 - \varepsilon$ there are $u, v \in M$ such that $\frac{f(u)-f(v)}{d(u,v)} > 1 - \varepsilon$ and $d(u,v) < \frac{\varepsilon}{(1-\varepsilon)^2} d(x,y)$.

Sketch of the proof: Apply Step 1 to the function $g = \frac{1}{4}\sum_{i=1}^4 f_i$ where $f_1 = f$, $f_2(t) = d(y,t)$, $f_3(t) = -d(x,t)$ and

$$f_4(t) = \frac{d(x,y)}{2}\frac{d(t,y) - d(t,x)}{d(t,y) + d(t,x)}$$

Assume that $\mathcal{F}(M)$ has the Daugavet property. Then M is local.

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Step 2. For every $f \in S_{\mathrm{Lip}_0(M)}$, $\varepsilon > 0$ and $x, y \in M$ such that $\frac{f(x)-f(y)}{d(x,y)} > 1 - \varepsilon$ there are $u, v \in M$ such that $\frac{f(u)-f(v)}{d(u,v)} > 1 - \varepsilon$ and $d(u,v) < \frac{\varepsilon}{(1-\varepsilon)^2} d(x,y)$.

Finally, given $f \in S_{\mathrm{Lip}_0(M)}$ use Step 2 to find sequences $(x_n), (y_n)$ with

$$d(x_n, y_n) < \left(\frac{\varepsilon}{(1-\varepsilon)^2}\right)^n d(x_0, y_0)$$

and $\frac{f(x_n)-f(y_n)}{d(x_n,y_n)} > 1 - \varepsilon$.

Strongly exposed points of $B_{\mathcal{F}(M)}$

Theorem (G.-L. – Procházka – Rueda-Zoca)

Let $x, y \in M$, $x \neq y$. The following are equivalent.

(i) $\frac{\delta_x - \delta_y}{d(x,y)}$ *is a strongly exposed point of* $B_{\mathcal{F}(M)}$.

(ii) *There is $f \in \mathrm{Lip}_0(M)$ peaking at (x, y), that is,*

$$\lim_n \frac{f(u_n) - f(v_n)}{d(u_n, v_n)} = 1 \Rightarrow \lim_n u_n = x, \lim_n v_n = y$$

(iii) *There is $\varepsilon > 0$ such that for every $z \in M \setminus \{x, y\}$,*

$$d(x,z) + d(y,z) > d(x,y) + \varepsilon \min\{d(x,z), d(y,z)\}$$

Corollary (G.-L. – Procházka – Rueda-Zoca)

Let M be a **compact** *metric space. Then $\mathrm{Lip}_0(M)$ has the Daugavet property if and only if $B_{\mathcal{F}(M)}$ does not have any strongly exposed point.*

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