Remarks on the set of norm-attaining functionals and differentiability

Antonio J. Guirao (Joint with V.Montesinos & V. Zizler)

IUMPA. Universitat Politècnica de València.

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Norm attaining functionals and BPB theorem

Given $f: X \to \mathbb{R}$ in $L(X,\mathbb{R}) = X^*$.

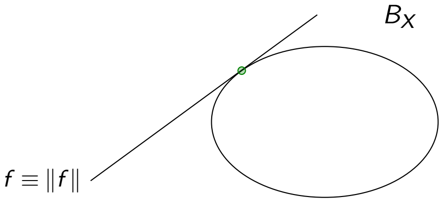
$$\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}$$

Definition: (Norm Attaining Operators)

A functional f is said a *norm attaining functional* when

$$\exists x_0 \in S_X \text{ such that } |f(x_0)| = \|f\|.$$

The set of those functionals is denoted by $\mathrm{NA}(X) \subset X^*$.



Norm attaining functionals and BPB theorem

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Theorem: (Bollobás, 1970)

For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x^*(x)| > 1 - \delta$ and $\|x^*\| = \|x\| = 1$ then, there exists x_0^* and x_0 ε-near of x^* and x such that $|x_0^*(x_0)| = 1 = \|x_0^*\| = \|x_0\|$. ($\delta$ universal)

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Ekeland VP+ Hahn-Banach separation.

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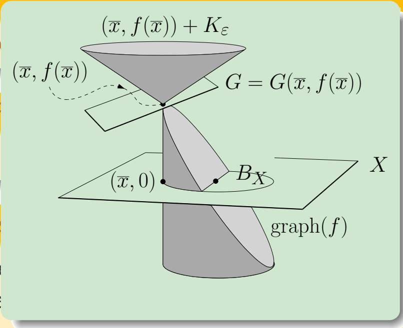

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Bishop-Phelps-Bollobás using differentiability.

Smooth Variational Principle (Deville-Godefroy-Zizler)

Theorem (Smooth Variational Principle)

$(X, \|\cdot\|)$ Banach. Assume:

- X has a G-diff. Lip. bump function b.
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X with a G-norm $\Rightarrow a = 1/8$.

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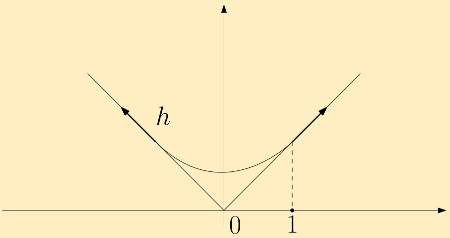

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$(X,\|\cdot\|)$ G-norm, then BPB Theorem with $\delta=\varepsilon^2/8$.

Steps of the proof:

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We don't use Hahn-Banach to separate, just Fermat theorem.

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The original norm can be any given norm! (and can be done $|\cdot| \overset{\delta}{\sim} \|\cdot\|$)

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- Set $z/|z| = x_0 + c$, then g attains its $\|\cdot\|^*$-norm at x_0. (so $\|x_0\| = 1$)

Corollary (BPB, WCG case)

X a WCG space. Then BPB Theorem with $\delta = \varepsilon^2/128$

Steps of the proof:

- Assume $f(x_\varepsilon) > 1 - \delta$.
- Find $|\cdot| \overset{\rho}{\sim} \|\cdot\|$ dual R, keeping norm attaining.
- Take $c_\varepsilon \in C$, $f(c_\varepsilon) = \sup_C f$. Set $z_\varepsilon = x_\varepsilon + c_\varepsilon \in B_{|\cdot|}$.
- $|f|^* = \|f\|^* + f(c_\varepsilon)$, so $f(z_\varepsilon) > |f|^* - \delta$.
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- Taking ρ small enough, $\|x_0 - x_\varepsilon\| < \varepsilon$ and $\|f - g\|^* < \varepsilon$.

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1. This is, somehow, a constructive version of BPB theorem: explict formula for the functional as the derivative of an explicit norm on a certain point. The point however, is still just a consequence of completeness.
2. Bishop–Phelps theorem needs a smaller δ.
3. The renorming trick does not work for "Fréchet differentiability": M.D. Acosta and M. Ruiz-Galán proved that
   - Every Banach space can be renormed so that the new set $NA(X)$ has an interior point.
   - $NA(X)$ has empty interior whenever the norm of a nonreflexive separable space X is Fréchet differentiable.

Residuality of $NA(X)$ and differentiability.

Definition: (Porous set)

$A \subset X$ is porous when $\exists \lambda \in (0,1)$:
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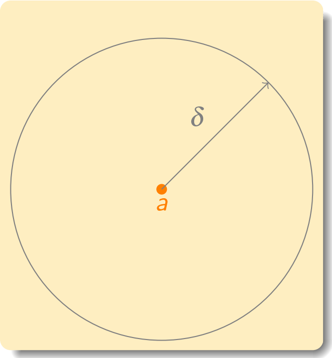

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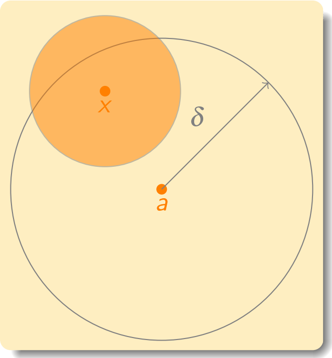

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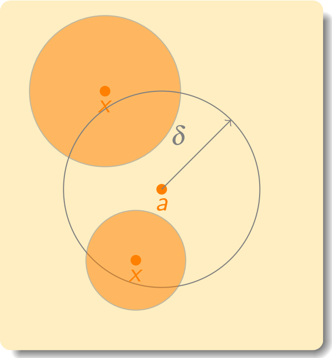

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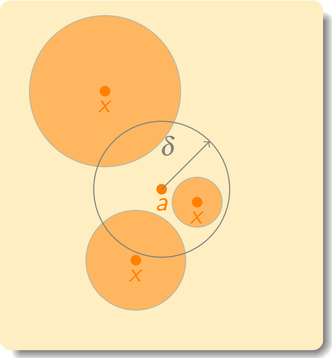

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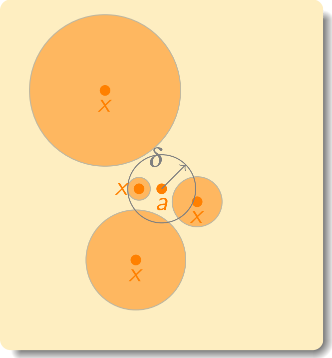

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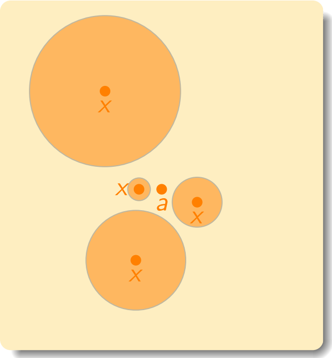

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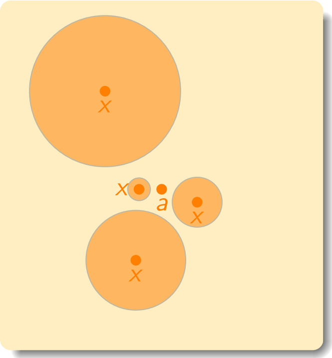

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- Bishop-Phelps theorem concludes that $F(X^*,\|\cdot\|^*)$ is dense.

Proof: [⇐]

$$F(X^*,\|\cdot\|^*) \subset NA(X,\|\cdot\|) \text{ is } G_\delta\text{-dense.}$$

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Remarks

1. Bourgain-Stegall: X separable, NA 2nd category $\Rightarrow B_X$ is dentable.
2. X LUR, then $NA(X) = F(X^*, \|\cdot\|^*) \Rightarrow NA(X)$ is G_δ-dense.

Thanks Tony and Matias!

Congratulations to Robert.

Thank you all for your attention!

The best behavior of norm attaining is under RNP

Proposition

X separable. TFAE:

- X has the RNP property.
- Every equivalent ball is dentable.
- Every equivalent G-ball is dentable.
- Every dual norm is Fréchet in a G_δ.
- Every equivalent norm, $NA(X)$ is residual.
- Every equivlaent norm, $NA(X)$ is 2nd category.

Definition

Every cbc set contains a point where $\tau_{\|\cdot\|} = w$.

Proposition

X separable. TFAE:

- X has the CPCP property.
- Every equivalent strictly convex norm on X is dentable.
- Every equivalent strictly convex norm on X, $NA(X)$ is residual.
- Every equivalent strictly convex norm on X, $NA(X)$ is of second category.

RNP implies CPCP. However, JT_* has CPCP but not the RNP