Lévy processes on Isometry C*-bialgebras

Jason Hancox

Lancaster University

Besançon
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We will call these the Isometry *-bialgebras and denote them $\mathcal{I}_0(d)$

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Lévy processes on a *-bialgebra

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A **convolution semigroup of states** $(\omega_t)_{t\geq 0}$ is a family of states such that

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Furthermore, every Schurmann triple on $\mathcal{I}_0(d)$ arises this way.

Toeplitz algebra i.e. $d = 1$

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- For this example we can extend the convolution semigroup of states associated to (V, h, λ) to the C*-algebra.

Lévy processes on the integer subalgebra

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Theorem

The Lévy process corresponding to (V, h, λ) restricts to a Lévy process on the circle if and only if V is unitary.

Brownian Motion on the Circle

Figure: $V = \mathrm{id}_{\mathbb{C}}, h = 1, \lambda = 0$

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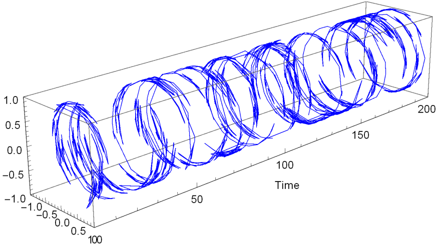

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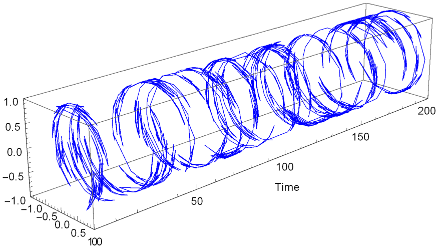

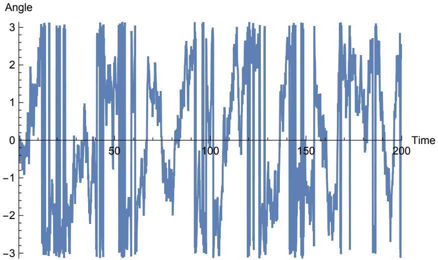

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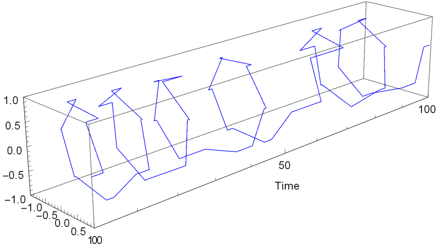

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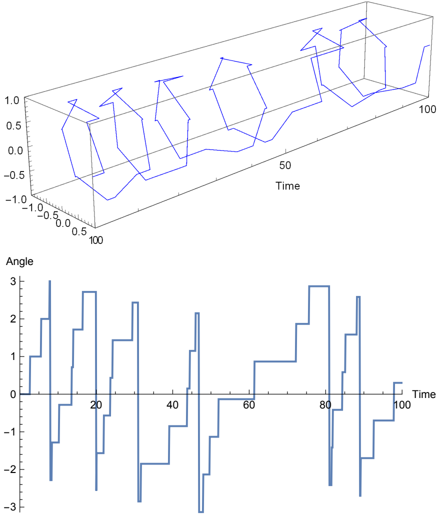

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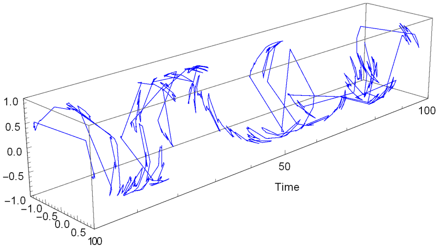

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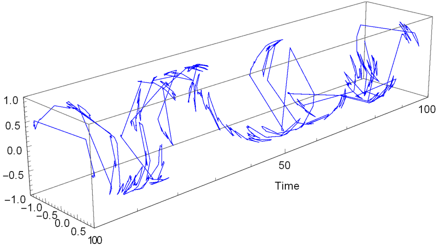

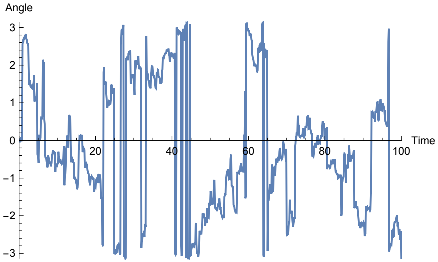

Fin.