On certain geometric properties in Banach spaces of vector-valued functions

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Köthe function spaces

Definition

Let $(S, \mathcal{A}, \mu)$ be a complete, σ-finite measure space and E a Banach space of (equivalence classes of) real-valued measurable functions on S. E is called a Köthe function space provided that:

(a) Every $f \in E$ is locally μ-integrable.

(b) $\chi_A \in E$ for all $A \in \mathcal{A}$ with $\mu(A) < \infty$.

(c) If $f \in E$ and g is measurable such that $|g(s)| \leq |f(s)|$ almost everywhere then $g \in E$ and $\|g\|_E \leq \|f\|_E$.

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$L^p(\mu)$ is a Köthe function space for all $1 \leq p \leq \infty$.

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Every Köthe function space is a Banach lattice in its natural ordering ($f \leq g \;:\Longleftrightarrow\; f(s) \leq g(s)$ a.e.).

Köthe-Bochner spaces

Definition

Let E be a Köthe function space over $(S, \mathcal{A}, \mu)$ and X a Banach space. Put

$$E(X) := \{f : S \to X : f \text{ is Bochner-measurable and } \|f(\cdot)\| \in E\}$$

and

$$\|f\|_{E(X)} := \|\|f(\cdot)\|\|_E \;\; \forall f \in E(X).$$

If one identifies two functions which are equal almost everywhere then $(E(X), \|\cdot\|)$ becomes a Banach space, the so called Köthe-Bochner space induced by E and X.

Associated finite sums

If E is a Köthe function space over $(S, \mathcal{A}, \mu)$, X a Banach space, and $A_1, \ldots, A_N \in \mathcal{A}$ are pairwise disjoint sets with $0 < \mu(A_i) < \infty$ for $i = 1, \ldots, N$, we define $E(A_1, \ldots, A_N, X)$ as the space X^N equppied with the norm

$$\|(x_1, \ldots, x_N)\|_{E(A_1,\ldots,A_N,X)} := \left\| \sum_{i=1}^{N} \frac{\|x_i\|}{\|\chi_{A_i}\|_E} \chi_{A_i} \right\|_E \ \forall (x_1, \ldots, x_N) \in X^N.$$

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If $p \in [1, \infty]$ and $E = L^p(\mu)$, then this space coincides with the usual N-fold p-direct sums of X, regardless of the choice of $A_1, \dots, A_N$.

General question

Consider a class $\mathcal{E}$ of Banach spaces which is closed under isometric isomorphisms.

Question

Given a Köthe function space E. If $\mathcal{E}$ is closed with respect to the formation of finite sums $E(A_1, \dots, A_N, X)$, can we conclude that $\mathcal{E}$ is also closed with respect to the formation of Köthe-Bochner spaces $E(X)$?

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Of course, in general the answer is "no".

For properties with a specific structure, the answer is "yes".

Notation

Definition

For a Banach space X, we denote by $\mathcal{U}(X)$ the set of all nontrivial, closed subspaces of X.
B_X denotes the closed unit ball and S_X the unit sphere of X.
For each $n \in \mathbb{N}$, B_X^n denotes the set of all sequences of length n in B_X, and $B_X^{\text{fin}} = \bigcup_{n=1}^{\infty} B_X^n$ denotes the set of all finite sequences in B_X.
Similar notations are used for S_X.

Test functions

Definition

A family $F_{\varepsilon,U} : B_U^{\text{fin}} \times B_U \times B_{U^*} \to \mathbb{R}$ indexed by $\varepsilon > 0$ and $U \in \mathcal{U}(X)$ is called a family of test functions for $\mathcal{E}$ in X if the following conditions are satisfied:

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(i) If $U \in \mathcal{U}(X)$, then: $U \in \mathcal{E} \Leftrightarrow \forall \varepsilon > 0 \; \forall \mathbf{x} \in S_U^{\text{fin}} \; \exists y \in S_U \; \exists y^* \in S_{U^*}$ such that $F_{\varepsilon,U}(\mathbf{x}, y, y^*) \leq \varepsilon$.

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(ii) $0 < \varepsilon_1 < \varepsilon_2 \Rightarrow F_{\varepsilon_1,U} \geq F_{\varepsilon_2,U}$.

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(ii) $0 < \varepsilon_1 < \varepsilon_2 \Rightarrow F_{\varepsilon_1,U} \geq F_{\varepsilon_2,U}$.

(iii) If $U \in \mathcal{U}(X)$, $\varepsilon > 0$, $\mathbf{x} \in B_U^{\text{fin}}$, $y \in B_U$ and $y^* \in B_{X^*}$, then $F_{\varepsilon,X}(\mathbf{x}, y, y^*) \leq F_{\varepsilon,U}(\mathbf{x}, y, y^*|_U)$.

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(iii) If $U \in \mathcal{U}(X)$, $\varepsilon > 0$, $\mathbf{x} \in B_U^{\text{fin}}$, $y \in B_U$ and $y^* \in B_{X^*}$, then $F_{\varepsilon,X}(\mathbf{x}, y, y^*) \leq F_{\varepsilon,U}(\mathbf{x}, y, y^*|_U)$.

(iv) $\forall \varepsilon > 0\ \forall n \in \mathbb{N}\ \forall \tau > 0\ \forall \mathbf{x} \in B_X^n\ \exists \delta > 0\ \forall y \in B_X\ \forall y^* \in B_{X^*}$: $\mathbf{z} \in B_X^n\ \|\mathbf{x} - \mathbf{z}\|_\infty \leq \delta \Rightarrow |F_{\varepsilon,X}(\mathbf{x}, y, y^*) - F_{\varepsilon,X}(\mathbf{z}, y, y^*)| \leq \tau$

Octahedrality

Definition (Haller, Langemets, Põldvere (2015))

(a) A Banach space X is called octahedral (OH) if the following holds: for all $x_1, \dots, x_n \in S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i + y\| \geq 2 - \varepsilon$ for $i = 1, \dots, n$.

(b) X is called locally octahedral (LOH) if the following holds: for every $x \in S_X$ and every $\varepsilon > 0$ there is some $y \in S_X$ such that $\|x \pm y\| \geq 2 - \varepsilon$.

Prototypical example of an OH space is ℓ^1.

Test functions for OH and LOH

For $U \in \mathcal{U}(X)$, $\varepsilon > 0$, $\mathbf{x} \in B_U^{\text{fin}}$, $y \in B_U$ and $y^* \in B_{U^*}$, let

$$F_{\varepsilon,U}(\mathbf{x}, y, y^*) := \max\{2 - \|x_i + y\| : i = 1, \dots, n\},$$

where n is the length of $\mathbf{x}$.
This defines a family of test functions for OH in X.

Test functions for OH and LOH

For $U \in \mathcal{U}(X)$, $\varepsilon > 0$, $\mathbf{x} \in B_U^{\text{fin}}$, $y \in B_U$ and $y^* \in B_{U^*}$, let

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This defines a family of test functions for OH in X.

By setting

$$F_{\varepsilon,U}(\mathbf{x}, y, y^*) := \max\{2 - \|x_1 + y\|, 2 - \|x_1 - y\|\}$$

we obtain a family of test functions for LOH in X.

Almost square spaces

Definition (Abrahamsen, Langemets, Lima (2016))

(a) A Banach space X is called almost square (ASQ) if the following holds: for all $x_1, \dots, x_n \in S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i - y\| \leq 1 + \varepsilon$ for $i = 1, \dots, n$.

(b) X is called locally almost square (LASQ) if the following holds: for every $x \in S_X$ and every $\varepsilon > 0$ there is some $y \in S_X$ such that $\|x \pm y\| \leq 1 + \varepsilon$.

Prototypical example of an ASQ space is c_0.

Test functions for ASQ and LASQ

For $U \in \mathcal{U}(X)$, $\varepsilon > 0$, $\mathbf{x} \in B_U^{\text{fin}}$, $y \in B_U$ and $y^* \in B_{U^*}$, let

$$F_{\varepsilon,U}(\mathbf{x}, y, y^*) := \max\{\|x_i - y\| - 1 : i = 1, \ldots, n\},$$

where n is the length of $\mathbf{x}$.
This defines a family of test functions for ASQ in X.

Test functions for ASQ and LASQ

For $U \in \mathcal{U}(X)$, $\varepsilon > 0$, $\mathbf{x} \in B_U^{\text{fin}}$, $y \in B_U$ and $y^* \in B_{U^*}$, let

$$F_{\varepsilon,U}(\mathbf{x}, y, y^*) := \max\{\|x_i - y\| - 1 : i = 1, \dots, n\},$$

where n is the length of $\mathbf{x}$.
This defines a family of test functions for ASQ in X.

By setting

$$F_{\varepsilon,U}(\mathbf{x}, y, y^*) := \max\{\|x_1 + y\| - 1, \|x_1 - y\| - 1\}$$

we get a family of test functions for LASQ in X.

Lushness

Definition (Boyko, Kadets, Martín, Werner (2007))

A Banach space X is called lush if the following holds: for all $x_1, x_2 \in S_X$ and every $\varepsilon > 0$ there exists a functional $y^* \in S_{X^*}$ such that $x_1 \in S(y^*, \varepsilon)$ and $\operatorname{dist}(x_2, \operatorname{aco}(S(y^*, \varepsilon))) < \varepsilon$, where $S(y^*, \varepsilon) := \{z \in B_X : y^*(z) > 1 - \varepsilon\}$ and aco denotes the absolutely convex hull.

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Examples of lush spaces are $C(K)$ for any compact Hausdorff space K and $L^1(\mu)$ for any σ-finite measure μ. Lushness is an important property since it ensures that the space has numerical index one.

Test functions for lushness

For $U \in \mathcal{U}(X)$, $\varepsilon > 0$, $\mathbf{x} \in B_U^{\text{fin}}$, $y \in B_U$ and $y^* \in B_{U^*}$, let

$$F_{\varepsilon,U}(\mathbf{x}, y, y^*) := \max\{1 - y^*(x_1), \text{dist}(x_2, \text{aco}(S(y^*, \varepsilon)))\}.$$

This defines a family of test functions for lushness in X.

Main Theorem

Theorem (H)

Let $(S, \mathcal{A}, \mu)$ be a complete, σ-finite measure space and E an order continuous Köthe function space over $(S, \mathcal{A}, \mu)$. Suppose that X is a Banach space such that $E(A_1, \ldots, A_N, X) \in \mathcal{E}$ for every $N \in \mathbb{N}$ and all pairwise disjoint sets $A_1, \ldots, A_N \in \mathcal{A}$ with $0 < \mu(A_i) < \infty$ for each i. Suppose further that there exists a family of test functions for $\mathcal{E}$ in $E(X)$. Then $E(X) \in \mathcal{E}$.
In particular, this holds for $E = L^p(\mu)$ if $1 \leq p < \infty$.

The proof is based on approximations by simple functions.

Version for L^∞

Theorem (H)

Let $(S, \mathcal{A}, \mu)$ be a complete, σ-finite measure space. Suppose that X is a Banach space such that $\ell^\infty_N(X) \in \mathcal{E}$ for all $N \in \mathbb{N}$ and $\ell^\infty(X) \in \mathcal{E}$. If there exists a family of test functions for $\mathcal{E}$ in $L^\infty(\mu, X)$, then $L^\infty(\mu, X) \in \mathcal{E}$.

Here the proof is based on approximations by measurable functions with countable range.

Applications

Lush spaces:

Known (Boyko, Kadets, Martín, Merí): If $(X_i)_{i \in I}$ is any family of lush spaces, then $\ell^1(X_i)$ and $\ell^\infty(X_i)$ are also lush.

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As a consequence we get:

Theorem

If $(S, \mathcal{A}, \mu)$ is any complete, σ-finite measure space and X is lush, then $L^1(\mu, X)$ and $L^\infty(\mu, X)$ are also lush.

This has been proved very recently by Kadets, Martín, Merí and Pérez by a different method.

Applications

OH and ASQ spaces:

Known (Haller, Langemets, Põldvere):
X or Y OH $\Rightarrow X \oplus_1 Y$ OH
X and Y OH $\Rightarrow X \oplus_\infty Y$ OH
X OH $\Rightarrow \ell^\infty(X)$

Known (Abrahamsen, Langemets, Lima):
X or Y ASQ $\Rightarrow X \oplus_\infty Y$ is ASQ
X ASQ $\Rightarrow \ell^\infty(X)$ ASQ

As a consequence we get:

Theorem

If $(S, \mathcal{A}, \mu)$ is any complete, σ-finite measure space and X is OH, then $L^1(\mu, X)$ and $L^\infty(\mu, X)$ are OH.
If X is ASQ, then $L^\infty(\mu, X)$ is ASQ.

Applications

LOH and LASQ spaces:

Known (Abrahamsen, Langemets, Lima):
LOH and LASQ are stable with respect to arbitrary (finite or infinite) absolute sums.

As a consequence we get:

Theorem

If $(S, \mathcal{A}, \mu)$ is any complete, σ-finite measure space and E an order continuous Köthe function space over $(S, \mathcal{A}, \mu)$, then $E(X)$ is LOH/LASQ whenever X is LOH/LASQ.
Also, $L^{\infty}(\mu, X)$ is LOH/LASQ whenever X is LOH/LASQ.