Approximately order zero maps between C*-algebras

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Banach Spaces & Optimization

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Order zero maps

Let A be a C*-algebra. We say that elements $a, b \in A$ are **orthogonal** if

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Let $\phi\colon A \to B$ be a completely positive (c.p.) map, that is, the induced map $\phi^{(n)}\colon \mathbb{M}_n(A) \to \mathbb{M}_n(B)$ acting between the matrix algebras is positive for each $n \in \mathbb{N}$.

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$$a \perp b \implies \phi(a) \perp \phi(b) \quad \text{for all } a, b \in A_+$$

(and then the same holds true for general elements $a, b \in A$).

Order zero maps

vs. disjointness preserving maps

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These two notions are very similar and thus there are similar representations theorem for both of them: roughly speaking, order zero and disjointness preserving maps are '*compressions*' of some $*$-homomorphism/Jordan $*$-homomorphism.

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Theorem A (Wolff, 1994)

Let A and B be unital C*-algebras and $\varphi\colon A \to B$ be a disjointness preserving map with $\varphi(\mathbf{1}_A) = \mathbf{1}_B$. Then, φ is a Jordan *-homomorphism.

For any subset C of a C*-algebra B, denote by C' its **commutant**, that is,

$$C' = \{a \in B\colon ab = ba \text{ for every } b \in C\}.$$

Let also $\mathscr{M}(B)$ stand for the **multiplier algebra** of B.

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Theorem A′ (Wolff, 1994)

Let A and B be C*-algebras with A being unital. Let also $\varphi\colon A \to B$ be a disjointness preserving map and set

$$\mathcal{C} = \overline{\varphi(\mathbf{1}_A)\{\varphi(\mathbf{1}_A)\}'}.$$

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Then:

(1) $\varphi(\mathcal{A}) \subseteq \mathcal{C}$;

(2) there exists a Jordan *-homomorphism $\psi\colon A \to \mathscr{M}(\mathcal{C})$ such that $\psi(\mathbf{1}_A) = \mathbf{1}_{\mathscr{M}(\mathcal{C})}$ and $\varphi(x) \equiv \varphi(\mathbf{1}_A)\psi(x)$.

W. Winter, J. Zacharias, *Completely positive maps of order zero*, Münster J. Math. **2** (2009), 311–324.

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Theorem B (Winter & Zacharias, 2009)

Let A and B be C*-algebras and $\varphi\colon A \to B$ a c.p. order zero map. Let also $\mathcal{C} = C^*(\varphi(A)) \subseteq B$ (the C*-algebra generated by the range of φ). Then, there exists a positive element $h \in \mathscr{M}(\mathcal{C}) \cap \mathcal{C}'$ satisfying $\|h\| = \|\varphi\|$ and a $*$-homomorphism

$$\pi\colon A \to \mathscr{M}(\mathcal{C}) \cap \{h\}'$$

such that

$$\varphi(a) = \pi(a)h \quad \text{for every } a \in A.$$

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such that

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Moreover, if A is unital, then we have $h = \varphi(\mathbf{1}_A) \in \mathcal{C}$.

Approximately order zero maps

Let A, B be C^*-algebras, $\phi\colon A \to B$ a bounded linear operator and $\varepsilon \geqslant 0$. We say that ϕ is an ε-*order zero map* (ε-o.z. for short) if it satisfies the condition

$$x, y \in A_+,\ x \perp y \implies \|\phi(x)\phi(y)\| \leqslant \varepsilon\|x\|\|y\|. \tag{1}$$

We say ϕ is ε-*self-adjoint* (ε-s.a. for short) if it satisfies

$$\|\phi(x^*) - \phi(x)^*\| \leqslant \varepsilon\|x\| \quad \text{for every } x \in A.$$

Finally, we call ϕ an ε-*disjointness preserving map* (ε-d.p. for short), provided it is both ε-o.z. and ε-s.a.

Ulam's stability problem

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This definition is exactly analogous to B.E. Johnson's definition of an 'AMNM *pair*' (of Banach algebras) which corresponds to approximately multiplicative maps.

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This definition is exactly analogous to B.E. Johnson's definition of an 'AMNM *pair*' (of Banach algebras) which corresponds to approximately multiplicative maps.

Main result

If A is a nuclear C^*-algebra and B is a von Neumann algebra, then (A, B) has the Ulam stability property for d.p. maps.

Stability in the commutative case

Let X, Y be compact Hausdorff spaces and $C(X)$, $C(Y)$ be the Banach spaces of continuous (complex valued) functions with the supremum norm (that is, commutative C*-algebras).

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- We say (as above) that φ is **disjointness preserving** if

$$fg = 0 \quad \text{implies} \quad \varphi(f)\varphi(g) = 0 \quad \text{for } f, g \in C(X).$$

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$$fg = 0 \quad \text{implies} \quad \varphi(f)\varphi(g) = 0 \quad \text{for } f, g \in C(X).$$

- We say that φ is **ε-disjointness preserving** if

$$fg = 0 \quad \text{implies} \quad \|\varphi(f)\varphi(g)\| \leqslant \varepsilon\|f\|\|g\| \quad \text{for } f, g \in C(X)$$

(where $\varepsilon \geqslant 0$ is a fixed number).

Stability in the commutative case

G. Dolinar, *Stability of disjointness preserving mappings*, Proc. Amer. Math. Soc. **130** (2001), 129–138.

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Theorem C (Dolinar, 2001)

Let X be a compact Hausdorff space and $\varepsilon \geqslant 0$. If φ is a continuous and ε-disjointness preserving linear functional on $C(X)$, then there exists a (continuous) disjointness preserving linear functional ψ on $C(X)$ such that

$$|\varphi(f) - \psi(f)| \leqslant 3\sqrt{\varepsilon}\|f\| \quad \text{for every } f \in C(X).$$

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By using the above theorem and the fact that we have an abundance of linear multiplicative functionals on $C(Y)$, Dolinar obtained also the following result:

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Theorem C′ (Dolinar, 2001)

Let X and Y be compact Hausdorff spaces and $\varepsilon \geqslant 0$. Let also $\varphi\colon C(X) \to C(Y)$ be an ε-disjointness preserving linear operator and assume that

- φ is continuous, or
- φ is surjective.

Then, there exists a disjointness preserving linear operator $\psi\colon C(X) \to C(Y)$ such that

$$\|\varphi(f) - \psi(f)\| \leqslant 20\sqrt{\varepsilon}\|f\| \quad \text{for every } f \in C(X).$$

Stability in the commutative case

An operator $\psi\colon C(X) \to C(Y)$ is called a *weighted composition map*, provided that there exist $\alpha \in C(Y)$ and a map $h\colon Y \to X$ which is continuous on the set $\{y \in Y\colon \alpha(y) \neq 0\}$ such that

$$\psi(f)(y) = \alpha(y)f(h(y)) \quad \text{for all } f \in C(X),\ y \in Y.$$

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The class of all weighted composition maps coincides with the class of all **continuous**, linear, disjointness preserving operators acting between $C(X)$ and $C(Y)$.

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J. Araujo, J.J. Font, *Stability of weighted composition operators between spaces of continuous functions*, J. London Math. Soc. **79** (2009), 363–376.

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Theorem D (Araujo & Font, 2009)

Let X and Y be compact Hausdorff spaces and let $0 < \varepsilon < \frac{2}{17}$. Then, for every ε-disjointness preserving linear operator $\varphi\colon C(X) \to C(Y)$ with $\|\varphi\| = 1$ there exists a weighted composition map $\psi\colon C(X) \to C(Y)$ such that

$$\|\varphi - \psi\| \leqslant \sqrt{\frac{17\varepsilon}{2}}\,.$$

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Moreover, for each $\varepsilon \in \big(0, \frac{2}{17}\big)$ there exists a norm one ε-disjointness preserving operator φ such that $\|\varphi - \psi\| \geqslant \sqrt{\frac{17\varepsilon}{2}}$ for every weighted composition map ψ.

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Moreover, for each $\varepsilon \in \left(0, \frac{2}{17}\right)$ there exists a norm one ε-disjointness preserving operator φ such that $\|\varphi - \psi\| \geqslant \sqrt{\frac{17\varepsilon}{2}}$ for every weighted composition map ψ. Similarly, there exists a norm one $\frac{2}{17}$-disjointness preserving operator φ such that $\|\varphi - \psi\| \geqslant 1$ for every weighted composition map ψ

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Moreover, for each $\varepsilon \in \left(0, \frac{2}{17}\right)$ there exists a norm one ε-disjointness preserving operator φ such that $\|\varphi - \psi\| \geqslant \sqrt{\frac{17\varepsilon}{2}}$ for every weighted composition map ψ. Similarly, there exists a norm one $\frac{2}{17}$-disjointness preserving operator φ such that $\|\varphi - \psi\| \geqslant 1$ for every weighted composition map ψ (hence, the case $\varepsilon \geqslant \frac{2}{17}$ is irrelevant).

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Araujo and Font also obtained the optimal stability result for ε-disjointness preserving functionals.

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First, note that if $\varphi \in C(X)^*$ and $\|\varphi\| = 1$, then ε-disjointness preserving is meaningless for $\varepsilon \geqslant \frac{1}{4}$ (in other words, every such φ is $\frac{1}{4}$-disjointness preserving).

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$$\|f + \alpha g\| = 1 \quad \text{for every } \alpha \in \mathbb{K} \text{ with } |\alpha| = 1,$$

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and hence $|\varphi(f) + \alpha\varphi(g)| \leqslant 1$ which implies $|\varphi(f)| + |\varphi(g)| \leqslant 1$.

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and hence $|\varphi(f) + \alpha\varphi(g)| \leqslant 1$ which implies $|\varphi(f)| + |\varphi(g)| \leqslant 1$. Therefore, $|\varphi(f)||\varphi(g)| \leqslant \frac{1}{4}$.

Stability in the commutative case

For each $n \in \mathbb{N}$ define

$$\omega_n = \frac{n^2 - 1}{4n^2} \quad \text{and} \quad \mathbb{A}_n = [\omega_{2n-1}, \omega_{2n+1}).$$

For any $\varepsilon \in [0, \frac{1}{4})$ and $n \in \mathbb{N}$ so that $\varepsilon \in \mathbb{A}_n$ define

$$\Theta_X(\varepsilon) = \begin{cases} \frac{2n-1-\sqrt{1-4\varepsilon}}{2n} & \text{if } 2n \leqslant \mathrm{card} X, \\ \frac{2m-1-\sqrt{1-4\varepsilon}}{2m} & \text{if } \mathrm{card} X = 2m < 2n, \\ \frac{2m-2}{2m-1} & \text{if } \mathrm{card} X = 2m-1 < 2n. \end{cases}$$

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Theorem D′ (Araujo & Font, 2010)

Let X be a compact Hausdorff space and $0 < \varepsilon < \frac{1}{4}$. Then, for every ε-disjointness preserving functional φ on $C(X)$ with $\|\varphi\| = 1$ there exists a weighted evaluation functional ψ on $C(X)$ (i.e. ψ corresponds to a multiple of some Dirac's measure) such that $\|\varphi - \psi\| \leqslant \Theta_X(\varepsilon)$.

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Stability in the non-commutative case
(for surjective maps)

J. Alaminos, J. Extremera, A.R. Villena, *Approximately zero product preserving maps*, Israel J. Math. **178** (2010), 1–28.

Theorem E (Alaminos, Extremera, Villena, 2010)

Let $\mathcal{A}$ be either the group algebra $L^1(G)$ for some l.c. group G or a C*-algebra and let $\mathcal{B}$ be a Banach algebra. Suppose that both $\mathcal{A}$ and $\mathcal{B}$ are amenable and that there is a Banach $\mathcal{B}$-bimodule X so that $\mathscr{M}(\mathcal{B})$ is isomorphic as a $\mathcal{B}$-bimodule with X^*.

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$$\|T(x)T(y)\| \leqslant \delta(\varepsilon)\|x\|\|y\| \quad \text{for all } x, y \in \mathcal{A} \text{ with } xy = 0,$$

Stability in the non-commutative case
(for surjective maps)

J. Alaminos, J. Extremera, A.R. Villena, *Approximately zero product preserving maps*, Israel J. Math. **178** (2010), 1–28.

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then there exist an invertible $\nu \in \mathscr{M}(\mathcal{B})'$ and a continuous, surjective homomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ such that $\|T - \nu\Phi\| \leqslant \varepsilon$.

Amenable Banach algebras

Nuclear C^*-algebras are exactly these which are **amenable** as Banach algebras—the concept introduced by B.E. Johnson in:

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(A non-cohomological) definiton of amenability (1)

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}\hat{\otimes}\mathcal{A}$ be its projective tensor product. By a **virtual diagonal** of $\mathcal{A}$ we mean any bounded net $(m_\alpha) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$ such that:

- $m_\alpha a - am_\alpha \to 0$ (in norm) for each $a \in \mathcal{A}$;
- $\pi(m_\alpha)a,\ a\pi(m_\alpha) \to a$ (in norm) for each $a \in \mathcal{A}$ (where $\pi\colon \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ is defined by $\pi(a \otimes b) = ab$).

(A non-cohomological) definiton of amenability (2)

"Equivalently", a virtual diagonal is any element $m \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ satisfying:

- $ma = am$ for each $a \in \mathcal{A}$;
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Remarks:

- $C(X)$ is always amenable for every compact Hausdorff space X;
- if G is an l.c. group, then $L_1(G)$ is amenable iff G is amenable;
- a C*-algebra is amenable iff it is nuclear (there are many examples of such).

Main result

If A is a nuclear C^*-algebra and B is a von Neumann algebra, then (A, B) has the Ulam stability property for d.p. maps.

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4. In general, there is a (corner) decomposition $\phi = \phi_{\mathrm{s}} + \phi_{\mathrm{r}}$, where $\|\phi_{\mathrm{s}}\| = \|\phi\|^{3/5} O(\varepsilon^{1/5})$ is small, $\phi_{\mathrm{r}}(1_A)$ is 'well-invertible' and ϕ_{r} is $\|\phi\|^{15/8}\varepsilon^{1/16}$-d.p.

General strategy of the proof

Step 2. We show that if $\phi \in \mathcal{L}(A, B)$ is ε-o.z., then it is an '*approximate Jordan homomorphism*' in the sense that

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By some standard techniques, we may work with positive x so that $\sigma(x) \subseteq [0, 1]$. We consider the generated C*-subalgebra $C(\sigma(x)) \cong C^*(x, \mathbf{1}_A)$, where x is identified with the identity function $\mathrm{id}_{\sigma(x)}$. Then, we estimate our difference approximating $\mathrm{id}_{\sigma(x)}$ by combinations of characteristic functions (on which T^{**} acts, as those functions are in the second dual of $C(\sigma(x))$).

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Now, when ϕ is unital this should imply that ϕ lies close to some Jordan homomorphism.

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and then the map $T \mapsto T + S$ gives the desired approximation procedure.