The isomorphism class of c_0 is not Borel

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Banach spaces and optimization:
Conference on the occasion of Robert Deville's 60th birthday
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Definition

The *space of separable Banach spaces* is defined as the set

$$\mathrm{SB} = \big\{F \subseteq C([0,1]) : F \text{ is closed and linear}\big\}$$

equipped with the Effros Borel structure, i.e., the σ-algebra generated by the sets

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Fact (Bossard)

The Effros Borel structure is the Borel σ-algebra of a Polish (i.e., separable completely metrizable) topology on SB.

General question

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Namely, what is the position of $\mathrm{SB} \cap \mathcal{C}$ in the projective hierarchy of classes of subsets of Polish spaces?

$$\begin{array}{ccccccccc}
 & & \textit{analytic} & & & & \mathbf{\Sigma}^1_2 & & \\
 & \rotatebox{-30}{$\subsetneq$} & & \rotatebox{30}{$\supsetneq$} & & \rotatebox{-30}{$\subsetneq$} & & \rotatebox{30}{$\supsetneq$} & \\
\textit{Borel} & & & & \mathbf{\Delta}^1_2 & & & & \mathbf{\Delta}^1_3 \quad \cdots \\
 & \rotatebox{30}{$\subsetneq$} & & \rotatebox{-30}{$\supsetneq$} & & \rotatebox{30}{$\subsetneq$} & & \rotatebox{-30}{$\supsetneq$} & \\
 & & \textit{coanalytic} & & & & \mathbf{\Pi}^1_2 & &
\end{array}$$

Notation:

- $X \simeq Y$ ... X is linearly isomorphic to Y,
- $X \hookrightarrow Y$ ... X is linearly isomorphic to some $Z \subseteq Y$,
- $\langle X \rangle = \{Y \in \mathrm{SB} : Y \simeq X\}$,
- $\mathrm{Sub}(X) = \{Y \in \mathrm{SB} : Y \hookrightarrow X\}$,
- c_{00} denotes the vector space of all sequences $x = \{x(n)\}_{n=1}^{\infty}$ of real numbers such that $x(n) \neq 0$ for finitely many n's only,
- $e_n = \mathbf{1}_{\{n\}}$ is the n-th canonical basic vector of c_{00},
- $\mathcal{P}(\mathbb{N})$ is the set of all subsets of $\mathbb{N}$ with its standard topology,
- $K(\mathcal{P}(\mathbb{N})) = \{K \subseteq \mathcal{P}(\mathbb{N}) : K \text{ compact}\}$ equipped with the Vietoris topology.

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Example 6 (K.)

The family of all separable spaces with the Schur property is $\mathbf{\Pi}^1_2$ but not $\mathbf{\Sigma}^1_2$.

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Theorem (Argyros, Gasparis, Motakis, 2016)

There exists an isomorphic predual of ℓ_1 with summable Szlenk index which is not isomorphic to c_0.

Employing the construction of Argyros, Gasparis and Motakis, we prove:

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$\langle c_0 \oplus X \rangle$ is not Borel if X satisfies at least one of these conditions:

- *X does not contain an isomorphic copy of c_0,*
- *X does not contain an infinite-dimensional reflexive subspace,*
- *X is a subspace of a space with an unconditional basis.*

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$\langle C(K) \rangle$ is not Borel for any infinite metrizable compact space K.

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The family of all spaces with an unconditional basis is not Borel.

Definition (original Tsirelson space)

Let $\|\cdot\|_{T^*}$ be the greatest norm on c_{00} such that $\|e_i\|_{T^*} = 1$ for every $i \in \mathbb{N}$ and

$$n < \mathrm{supp}\ x_1 < \mathrm{supp}\ x_2 < \ldots < \mathrm{supp}\ x_n$$

$$\Rightarrow \quad \Big\|\sum_{k=1}^{n} x_k\Big\|_{T^*} \leq 2 \sup_{1 \leq k \leq n} \|x_k\|_{T^*}$$

whenever $n \in \mathbb{N}$ and $x_k \in c_{00}, k = 1, 2, \ldots, n$.

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Let T^* be the completion of $(c_{00}, \|\cdot\|_{T^*})$.

Definition (Argyros, Deliyanni)

Suppose that $\mathcal{M} \in K(\mathcal{P}(\mathbb{N}))$. Let $\|\cdot\|_{\mathcal{M}}$ be the greatest norm on c_{00} such that $\|e_i\|_{\mathcal{M}} = 1$ for every $i \in \mathbb{N}$ and

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Lemma (Argyros, Deliyanni)

If $\mathcal{M}$ consists of finite sets only, then $e_1, e_2, \ldots$ is a boundedly complete basis of $T^*[\mathcal{M}, \frac{1}{2}]$, and so $T^*[\mathcal{M}, \frac{1}{2}] \not\hookrightarrow c_0$.

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Lemma

If $\mathcal{M}$ contains an infinite set, then $T^*[\mathcal{M}, \frac{1}{2}]$ is isomorphic to the c_0-sum of a sequence of finite-dimensional spaces, and so $T^*[\mathcal{M}, \frac{1}{2}] \hookrightarrow c_0$.

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Proof (sketch)

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$$\mathfrak{H} = \Big\{ \mathcal{M} \in K(\mathcal{P}(\mathbb{N})) : \mathcal{M} \text{ contains an infinite set} \Big\}$$

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It can be shown that there exists a Borel mapping $\mathfrak{S} : K(\mathcal{P}(\mathbb{N})) \to \mathrm{SB}$ such that $\mathfrak{S}(\mathcal{M})$ is isometric to $T^*[\mathcal{M}, \frac{1}{2}]$ for every $\mathcal{M} \in K(\mathcal{P}(\mathbb{N}))$.

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Then we obtain $\mathfrak{H} = \mathfrak{S}^{-1}(\mathrm{Sub}(c_0))$, and Theorem 0 follows.

Definition

An infinite-dimensional Banach space X is called a $\mathcal{L}_{\infty,\lambda}$-*space* if, for any finite-dimensional subspace F of X, there exists a finite-dimensional subspace G of X containing F such that $d_{BM}(G, \ell_\infty^n) \leq \lambda$, where $n = \dim G$.

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If X is a $\mathcal{L}_\infty$-space and $X \hookrightarrow c_0$, then $X \simeq c_0$.

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Idea of the proof of Theorem 1

- Use the same method as above, but construct $\mathcal{L}_\infty$-spaces.
- The space of Argyros, Gasparis and Motakis plays the role of the (original) Tsirelson space now.

Proof of Theorem 1 (sketch)

Given $\mathcal{M} \in K(\mathcal{P}(\mathbb{N}))$, we construct a separable Banach space $\mathfrak{X}_\mathcal{M}$ such that

- $\mathfrak{X}_\mathcal{M}$ is a $\mathcal{L}_\infty$-space,
- $\mathfrak{X}_\mathcal{M}$ has a finite-dimensional decomposition $x = \sum_{i=0}^\infty P_i x$ such that

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Then we obtain

$$\mathfrak{X}_\mathcal{M} \simeq c_0 \quad \Leftrightarrow \quad \mathcal{M} \in \mathfrak{H},$$

and Theorem 1 follows in the same way as Theorem 0.

As well as the construction of the example of Argyros, Gasparis and Motakis, the construction of $\mathfrak{X}_{\mathcal{M}}$ is based on a method of Bourgain and Delbaen:

A construction of a $\mathcal{L}_\infty$-space (part 1 of 2)

- $\Gamma_0 \subseteq \Gamma_1 \subseteq \dots$ is a suitable increasing sequence of finite sets,
- $c^*_\gamma \in (\ell_\infty(\Gamma_{p-1}))^*$ is chosen for every $p \geq 1$ and $\gamma \in \Gamma_p \setminus \Gamma_{p-1}$,
- $i_{p-1,p} : \ell_\infty(\Gamma_{p-1}) \to \ell_\infty(\Gamma_p)$ is the extension operator

$$i_{p-1,p}(x)(\gamma) = \begin{cases} x(\gamma), & \gamma \in \Gamma_{p-1}, \\ c^*_\gamma(x), & \gamma \in \Gamma_p \setminus \Gamma_{p-1}, \end{cases}$$

- for $0 \leq p < q$, we define $i_{p,q} = i_{q-1,q} \circ \dots \circ i_{p,p+1} : \ell_\infty(\Gamma_p) \to \ell_\infty(\Gamma_q)$.

A construction of a $\mathcal{L}_\infty$-space (part 2 of 2)

If the requirement

$$\sup_{0\leq p<q} \|i_{p,q}\| < \infty$$

is satisfied, then we can define:

- $\Gamma = \bigcup_{p=0}^{\infty} \Gamma_p$,
- $i_p : \ell_\infty(\Gamma_p) \to \ell_\infty(\Gamma)$ for $p \geq 0$ given by

$$i_p(x)(\gamma) = \begin{cases} x(\gamma), & \gamma \in \Gamma_p, \\ i_{p,q}(x)(\gamma), & \gamma \in \Gamma_q \setminus \Gamma_{q-1}, q > p, \end{cases}$$

- $d_\gamma = i_p(e_\gamma) \in \ell_\infty(\Gamma)$ for $p \geq 0$ and $\gamma \in \Gamma_p \setminus \Gamma_{p-1}$,
- $\mathfrak{X}_{\mathcal{M}} = \overline{\mathrm{span}}\,\{d_\gamma : \gamma \in \Gamma\}$,
- $P_{[0,p]} = i_p \circ r_p$, where $r_p : \mathfrak{X}_{\mathcal{M}} \to \ell_\infty(\Gamma_p)$ is the corresponding restriction operator.

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If the sets Γ_p and the functionals c_γ^* are chosen properly, then the construction works.

S. A. Argyros and I. Deliyanni, *Examples of asymptotic ℓ_1 Banach spaces*, Trans. Amer. Math. Soc. **349**, no. 3 (1997), 973–995.

S. A. Argyros, I. Gasparis and P. Motakis, *On the structure of separable $\mathcal{L}_\infty$-spaces*, Mathematika **62**, no. 3 (2016), 685–700.

B. Bossard, *A coding of separable Banach spaces. Analytic and coanalytic families of Banach spaces*, Fund. Math. **172**, no. 2 (2002), 117–152.

J. Bourgain and F. Delbaen, *A class of special $\mathcal{L}_\infty$ spaces*, Acta Math. **145**, no. 1 (1980), 155–176.

G. Godefroy, *Analytic sets of Banach spaces*, Rev. R. Acad. Cien. Serie A. Mat. **104**, no. 2 (2010), 365–374.

G. Godefroy, *The isomorphism classes of ℓ_p are Borel*, Houston J. Math. **43**, no. 3 (2017), 947–951.

G. Godefroy, N. J. Kalton and G. Lancien, *Szlenk indices and uniform homeomorphisms*, Trans. Amer. Math. Soc. **353** (2001), 3895–3918.

W. B. Johnson and M. Zippin, *On subspaces of quotients of $(\sum G_n)_{\ell_p}$ and $(\sum G_n)_{c_0}$*, Israel J. Math. **13**, no. 3 (1972), 311–316.

O. Kurka, *Tsirelson-like spaces and complexity of classes of Banach spaces*, Rev. R. Acad. Cien. Serie A. Mat. (to appear).