Tsirelson-like spaces and complexity questions in Banach space theory

Ondřej Kurka

Charles University and Czech Academy of Sciences

Septièmes journées “Besançon-Neuchâtel” d’Analyse Fonctionnelle
Besançon, June 21, 2017

Definition

A family $\{E_1, \ldots, E_n\}$ of succesive finite subsets of $\mathbb{N}$ is said to be *admissible* if

$$n < E_1 < E_2 < \cdots < E_n.$$

The system of all admissible families is denoted by adm.

Definition

A family $\{E_1, \dots, E_n\}$ of succesive finite subsets of $\mathbb{N}$ is said to be *admissible* if

$$n < E_1 < E_2 < \cdots < E_n.$$

The system of all admissible families is denoted by adm.

Definition (Tsirelson, 1974)

Let Θ be the smallest symmetric convex subset of c_{00} containing every unit basic vector $e_i = \mathbf{1}_{\{i\}}$, $i \in \mathbb{N}$, and satisfying

$$\{E_1, \dots, E_n\} \in \mathrm{adm} \;\&\; x_1, \dots, x_n \in \Theta \quad \Rightarrow \quad \frac{1}{2} \sum_{k=1}^{n} E_k x_k \in \Theta.$$

Definition

A family $\{E_1, \ldots, E_n\}$ of succesive finite subsets of $\mathbb{N}$ is said to be *admissible* if

$$n < E_1 < E_2 < \cdots < E_n.$$

The system of all admissible families is denoted by adm.

Definition (Tsirelson, 1974)

Let Θ be the smallest symmetric convex subset of c_{00} containing every unit basic vector $e_i = \mathbf{1}_{\{i\}}$, $i \in \mathbb{N}$, and satisfying

$$\{E_1, \ldots, E_n\} \in \mathrm{adm} \;\&\; x_1, \ldots, x_n \in \Theta \quad \Rightarrow \quad \frac{1}{2}\sum_{k=1}^{n} E_k x_k \in \Theta.$$

Let $\|\cdot\|_{T^*}$ be the Minkowski gauge of Θ and let T^* be a completion of $(c_{00}, \|\cdot\|_{T^*})$.

Remark

T^* was the first example of an infinite-dimensional Banach space such that $c_0 \not\hookrightarrow T^*$ and $\ell_p \not\hookrightarrow T^*$ for $1 \leq p < \infty$.

Remark

T^* was the first example of an infinite-dimensional Banach space such that $c_0 \not\hookrightarrow T^*$ and $\ell_p \not\hookrightarrow T^*$ for $1 \leq p < \infty$.

Remark

T^* is reflexive and dual to the space T defined as the Banach space of sequences $x = \{x(i)\}_{i=1}^{\infty}$ with the unit vector basis and with the implicitly defined norm

$$\|x\|_T = \max\left\{\|x\|_\infty, \sup_{\{E_1,\ldots,E_n\}\in\mathrm{adm}} \frac{1}{2}\sum_{k=1}^{n}\|E_k x\|_T\right\}.$$

Definition (Argyros, Deliyanni)

For $\mathcal{M} \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$, a family $\{E_1, \ldots, E_n\}$ of succesive finite subsets of $\mathbb{N}$ is said to be $\mathcal{M}$-*admissible* if

$$\exists \nu \in \mathcal{M}\ \exists m_1, \ldots, m_n \in \mathbb{N} : \nu(m_1) = 1, \ldots, \nu(m_n) = 1$$
$$\& \ m_1 \leq E_1 < m_2 \leq E_2 < \cdots < m_n \leq E_n.$$

Definition (Argyros, Deliyanni)

For $\mathcal{M} \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$, a family $\{E_1, \ldots, E_n\}$ of succesive finite subsets of $\mathbb{N}$ is said to be $\mathcal{M}$-*admissible* if

$$\exists \nu \in \mathcal{M}\ \exists m_1, \ldots, m_n \in \mathbb{N} : \nu(m_1) = 1, \ldots, \nu(m_n) = 1$$
$$\& \ m_1 \leq E_1 < m_2 \leq E_2 < \cdots < m_n \leq E_n.$$

The *Tsirelson-like space* $T^*[\mathcal{M}, \frac{1}{2}]$ associated to $\mathcal{M} \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$ is defined in the same way as the Tsirelson space with the difference that we consider $\mathcal{M}$-admissible families instead of admissible ones.

Definition (Argyros, Deliyanni)

For $\mathcal{M} \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$, a family $\{E_1, \ldots, E_n\}$ of succesive finite subsets of $\mathbb{N}$ is said to be $\mathcal{M}$-*admissible* if

$$\exists \nu \in \mathcal{M}\ \exists m_1, \ldots, m_n \in \mathbb{N} : \nu(m_1) = 1, \ldots, \nu(m_n) = 1 \\ \& \ m_1 \leq E_1 < m_2 \leq E_2 < \cdots < m_n \leq E_n.$$

The *Tsirelson-like space* $T^*[\mathcal{M}, \frac{1}{2}]$ associated to $\mathcal{M} \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$ is defined in the same way as the Tsirelson space with the difference that we consider $\mathcal{M}$-admissible families instead of admissible ones.

Lemma (Argyros, Deliyanni)

If $\mathcal{M}$ consists of sequences with finitely many 1's, then $e_1, e_2, \ldots$ is a boundedly complete basis of $T^*[\mathcal{M}, \frac{1}{2}]$, and so $T^*[\mathcal{M}, \frac{1}{2}] \not\hookrightarrow c_0$.

Lemma

If $\mathcal{M}$ contains a sequence with infinitely many 1's, then $T^*[\mathcal{M}, \frac{1}{2}] \hookrightarrow c_0$.

Lemma

If $\mathcal{M}$ contains a sequence with infinitely many 1's, then $T^*[\mathcal{M}, \frac{1}{2}] \hookrightarrow c_0$.

Proof.

If $\nu(m_1) = \nu(m_2) = \cdots = 1$ for some $\nu \in \mathcal{M}$, then

$$\sup_{k \in \mathbb{N} \cup \{0\}} \|E_k x\|_{\mathcal{M}} \leq \|x\|_{\mathcal{M}} \leq \|E_0 x\|_{\mathcal{M}} + 2 \sup_{k \in \mathbb{N}} \|E_k x\|_{\mathcal{M}}, \quad x \in T^*\Big[\mathcal{M}, \frac{1}{2}\Big],$$

where $E_0 = \{1, \ldots, m_1 - 1\}$ and $E_k = \{m_k, \ldots, m_{k+1} - 1\}$, hence the space $T^*[\mathcal{M}, \frac{1}{2}]$ is isomorphic to the c_0-sum of a sequence of finite-dimensional spaces. □

First main result (unprecisely)

The property $X \hookrightarrow c_0$ is as complex as it seems to be.

First main result (unprecisely)

The property $X \hookrightarrow c_0$ is as complex as it seems to be.

Reason: The property $X \hookrightarrow c_0$ is at least as complex as the property of a closed subset $\mathcal{M}$ of $\{0,1\}^{\mathbb{N}}$ that it contains a sequence with infinitely many 1's.

First main result (unprecisely)

The property $X \hookrightarrow c_0$ is as complex as it seems to be.

Reason: The property $X \hookrightarrow c_0$ is at least as complex as the property of a closed subset $\mathcal{M}$ of $\{0, 1\}^{\mathbb{N}}$ that it contains a sequence with infinitely many 1's.

Second main result (unprecisely)

The Schur property and the Dunford-Pettis property are as complex as they seem to be.

First main result (unprecisely)

The property $X \hookrightarrow c_0$ is as complex as it seems to be.

Reason: The property $X \hookrightarrow c_0$ is at least as complex as the property of a closed subset $\mathcal{M}$ of $\{0,1\}^{\mathbb{N}}$ that it contains a sequence with infinitely many 1's.

Second main result (unprecisely)

The Schur property and the Dunford-Pettis property are as complex as they seem to be.

Reason: It is possible to use tree spaces constructed from Tsirelson-like spaces.

Definition

A *Polish space (topology)* means a separable completely metrizable space (topology). A set P equipped with a σ-algebra is called a *standard Borel space* if the σ-algebra is generated by a Polish topology on P.

Definition

A *Polish space (topology)* means a separable completely metrizable space (topology). A set P equipped with a σ-algebra is called a *standard Borel space* if the σ-algebra is generated by a Polish topology on P.
A subset A of a standard Borel space X is called an *analytic set* (or a $\mathbf{\Sigma}_1^1$ *set*) if there exist a standard Borel space Y and a Borel subset B of $X \times Y$ such that A is the projection of B on the first coordinate. The complement of an analytic set is called a *coanalytic set* (or a $\mathbf{\Pi}_1^1$ *set*).

Definition

A *Polish space (topology)* means a separable completely metrizable space (topology). A set P equipped with a σ-algebra is called a *standard Borel space* if the σ-algebra is generated by a Polish topology on P.
A subset A of a standard Borel space X is called an *analytic set* (or a $\boldsymbol{\Sigma}^1_1$ *set*) if there exist a standard Borel space Y and a Borel subset B of $X \times Y$ such that A is the projection of B on the first coordinate. The complement of an analytic set is called a *coanalytic set* (or a $\boldsymbol{\Pi}^1_1$ *set*).
A subset A of a standard Borel space X is called a $\boldsymbol{\Sigma}^1_2$ *set* if there exist a standard Borel space Y and a coanalytic subset B of $X \times Y$ such that A is the projection of B on the first coordinate. The complement of a $\boldsymbol{\Sigma}^1_2$ set is called a $\boldsymbol{\Pi}^1_2$ *set*.

Definition

The *space of separable Banach spaces* is defined as the set

$$\mathrm{SB} = \{F \subseteq C([0,1]) : F \text{ is closed and linear}\}$$

equipped with the Effros Borel structure, i.e., the σ-algebra generated by the sets

$$\{F \in \mathrm{SB} : F \cap U \neq \emptyset\}$$

where U varies over open subsets of $C([0,1])$.

Definition

The *space of separable Banach spaces* is defined as the set

$$\mathrm{SB} = \{F \subseteq C([0,1]) : F \text{ is closed and linear}\}$$

equipped with the Effros Borel structure, i.e., the σ-algebra generated by the sets

$$\{F \in \mathrm{SB} : F \cap U \neq \emptyset\}$$

where U varies over open subsets of $C([0,1])$.

Fact

SB *is a standard Borel space.*

Definition

The *space of separable Banach spaces* is defined as the set

$$\mathrm{SB} = \{F \subseteq C([0,1]) : F \text{ is closed and linear}\}$$

equipped with the Effros Borel structure, i.e., the σ-algebra generated by the sets

$$\{F \in \mathrm{SB} : F \cap U \neq \emptyset\}$$

where U varies over open subsets of $C([0,1])$.

Fact

SB *is a standard Borel space.*

Therefore, it makes sense to say that a class of separable Banach spaces is Borel, analytic, coanalytic, $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_2$, ...

Example 1

For any separable Banach space Z, the class of all spaces isometric to Z is Borel.

Example 1

For any separable Banach space Z, the class of all spaces isometric to Z is Borel.

Example 2

For $1 < p < \infty$, the class of all spaces isomorphic to ℓ_p is Borel.

Example 1

For any separable Banach space Z, the class of all spaces isometric to Z is Borel.

Example 2

For $1 < p < \infty$, the class of all spaces isomorphic to ℓ_p is Borel.

Example 3

For $1 < p < \infty, p \neq 2$, the class of all spaces isomorphic to L_p is analytic but not Borel.

Example 1

For any separable Banach space Z, the class of all spaces isometric to Z is Borel.

Example 2

For $1 < p < \infty$, the class of all spaces isomorphic to ℓ_p is Borel.

Example 3

For $1 < p < \infty, p \neq 2$, the class of all spaces isomorphic to L_p is analytic but not Borel.

Example 4

The class of all separable reflexive spaces is coanalytic but not Borel.

Example 1

For any separable Banach space Z, the class of all spaces isometric to Z is Borel.

Example 2

For $1 < p < \infty$, the class of all spaces isomorphic to ℓ_p is Borel.

Example 3

For $1 < p < \infty, p \neq 2$, the class of all spaces isomorphic to L_p is analytic but not Borel.

Example 4

The class of all separable reflexive spaces is coanalytic but not Borel.

Example 5

It is not known if the class of spaces with a Schauder basis is Borel.

Question (Godefroy)

Is there any separable Banach space Z such that the class of Banach spaces which can be embedded into Z is not Borel? What about c_0?

Question (Godefroy)

Is there any separable Banach space Z such that the class of Banach spaces which can be embedded into Z is not Borel? What about c_0?

Theorem 1

The class of all spaces which can be embedded into c_0 is analytic but not Borel.

Question (Godefroy)

Is there any separable Banach space Z such that the class of Banach spaces which can be embedded into Z is not Borel? What about c_0?

Theorem 1

The class of all spaces which can be embedded into c_0 is analytic but not Borel.

Theorem 2

The class of all separable spaces with the Schur property is $\mathbf{\Pi}^1_2$ but not $\mathbf{\Sigma}^1_2$.
The same result holds for the Dunford-Pettis property.

Theorem (Hurewicz)

The set

$$\left\{\mathcal{M} \in \mathcal{K}(\{0,1\}^{\mathbb{N}}) : \mathcal{M} \text{ contains a sequence with infinitely many 1's}\right\}$$

is analytic but not Borel.

(On $\mathcal{K}(\{0,1\}^{\mathbb{N}})$, the topology generated by the Hausdorff metric is considered.)

Theorem (Hurewicz)

The set

$$\Big\{\mathcal{M} \in \mathcal{K}(\{0,1\}^{\mathbb{N}}) : \mathcal{M} \text{ contains a sequence with infinitely many 1's}\Big\}$$

is analytic but not Borel.

(On $\mathcal{K}(\{0,1\}^{\mathbb{N}})$, the topology generated by the Hausdorff metric is considered.)

Lemma

There exists a Borel mapping $\mathfrak{S} : \mathcal{K}(\{0,1\}^{\mathbb{N}}) \to \mathrm{SB}$ *such that* $\mathfrak{S}(\mathcal{M})$ *is isometric to* $T^*[\mathcal{M}, \frac{1}{2}]$ *for every* $\mathcal{M} \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$.

Theorem (Hurewicz)

The set

$$\Big\{\mathcal{M} \in \mathcal{K}(\{0,1\}^{\mathbb{N}}) : \mathcal{M} \text{ contains a sequence with infinitely many 1's}\Big\}$$

is analytic but not Borel.

(On $\mathcal{K}(\{0,1\}^{\mathbb{N}})$, the topology generated by the Hausdorff metric is considered.)

Lemma

There exists a Borel mapping $\mathfrak{S} : \mathcal{K}(\{0,1\}^{\mathbb{N}}) \to \mathrm{SB}$ *such that* $\mathfrak{S}(\mathcal{M})$ *is isometric to* $T^*[\mathcal{M}, \frac{1}{2}]$ *for every* $\mathcal{M} \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$.

Proof of Theorem 1.

... □

For $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$ and $\sigma \in 2^{\mathbb{N}}$, let us denote

$$\mathfrak{T}(\sigma) \in \mathrm{Tr}(\mathbb{N}), \quad \mathfrak{T}(\sigma) = \{\nu = (n_1, n_2, \ldots, n_\ell) \in \mathbb{N}^{<\mathbb{N}} : (\sigma|_\ell, \nu) \in \mathfrak{T}\}.$$

For $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$ and $\sigma \in 2^{\mathbb{N}}$, let us denote

$$\mathfrak{T}(\sigma) \in \mathrm{Tr}(\mathbb{N}), \quad \mathfrak{T}(\sigma) = \{\nu = (n_1, n_2, \dots, n_\ell) \in \mathbb{N}^{<\mathbb{N}} : (\sigma|_\ell, \nu) \in \mathfrak{T}\}.$$

Lemma

The set

$$\{\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N}) : (\forall \sigma \in 2^{\mathbb{N}})(\mathfrak{T}(\sigma) \in \mathrm{IF})\}$$

is a $\mathbf{\Pi}^1_2$ *but not* $\mathbf{\Sigma}^1_2$ *subset of* $\mathrm{Tr}(2 \times \mathbb{N})$.

For $\nu = (n_1, n_2, \ldots, n_\ell) \in \mathbb{N}^{<\mathbb{N}}$, let
$\tilde{\nu} = \{n_1, n_1 + n_2, \ldots, \sum_{i=1}^{\ell} n_i\} \subseteq \mathbb{N}$.
Analogously, for $\nu = (n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$, let
$\tilde{\nu} = \{n_1, n_1 + n_2, \ldots\} \subseteq \mathbb{N}$.

For $\nu = (n_1, n_2, \ldots, n_\ell) \in \mathbb{N}^{<\mathbb{N}}$, let
$\tilde{\nu} = \{n_1, n_1 + n_2, \ldots, \sum_{i=1}^{\ell} n_i\} \subseteq \mathbb{N}$.
Analogously, for $\nu = (n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$, let
$\tilde{\nu} = \{n_1, n_1 + n_2, \ldots\} \subseteq \mathbb{N}$.

For $\mathcal{T} \in \mathrm{Tr}(\mathbb{N})$, let $\mathcal{M}_{\mathcal{T}} = \{\tilde{\nu} : \nu \in \mathcal{T} \cup [\mathcal{T}] \text{ or } |\nu| \leq 3\}$
(where we identify subsets of $\mathbb{N}$ with elements of $2^{\mathbb{N}}$).

Lemma

If $\mathcal{T}$ is well-founded, then $T^*[\mathcal{M}_{\mathcal{T}}, \frac{1}{2}]$ is reflexive.

Lemma

If $\mathcal{T}$ *is well-founded, then* $T^*[\mathcal{M}_{\mathcal{T}}, \frac{1}{2}]$ *is reflexive.*

Lemma

If $\mathcal{T}$ *is ill-founded, then* $T^*[\mathcal{M}_{\mathcal{T}}, \frac{1}{2}]$ *is isomorphic to the* c_0*-sum of a sequence of finite-dimensional spaces.*

Lemma

If $\mathcal{T}$ is well-founded, then $T^*[\mathcal{M}_{\mathcal{T}}, \frac{1}{2}]$ is reflexive.

Lemma

If $\mathcal{T}$ is ill-founded, then $T^*[\mathcal{M}_{\mathcal{T}}, \frac{1}{2}]$ is isomorphic to the c_0-sum of a sequence of finite-dimensional spaces.

Remark

If $\mathcal{T}$ is ill-founded, then the dual of $T^*[\mathcal{M}_{\mathcal{T}}, \frac{1}{2}]$ is isomorphic to the ℓ_1-sum of a sequence of finite-dimensional spaces. In particular, the dual of $T^*[\mathcal{M}_{\mathcal{T}}, \frac{1}{2}]$ has the Schur property.

Fact

If $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$ *and* $\sigma, \tau \in 2^{\mathbb{N}}$ *satisfy* $\sigma|_\ell = \tau|_\ell$*, then* $\|x\|_{\mathcal{M}_{\mathfrak{T}(\sigma)}} = \|x\|_{\mathcal{M}_{\mathfrak{T}(\tau)}}$ *for every* $x \in \mathrm{span}\{e_1, e_2, \ldots, e_\ell\}$*.*

Fact

If $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$ *and* $\sigma, \tau \in 2^{\mathbb{N}}$ *satisfy* $\sigma|_\ell = \tau|_\ell$*, then* $\|x\|_{\mathcal{M}_{\mathfrak{T}(\sigma)}} = \|x\|_{\mathcal{M}_{\mathfrak{T}(\tau)}}$ *for every* $x \in \mathrm{span}\{e_1, e_2, \ldots, e_\ell\}$*.*

Definition

For $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$, let $E_{\mathfrak{T}}$ be the completion of $c_{00}(2^{<\mathbb{N}} \setminus \{\emptyset\})$ with the norm

$$\|x\|_{\mathfrak{T}} = \sup_{\sigma \in 2^{\mathbb{N}}} \left\| \sum_{\ell=1}^{\infty} x(\sigma|_\ell) e_\ell \right\|_{\mathcal{M}_{\mathfrak{T}(\sigma)}}.$$

Fact

If $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$ *and* $\sigma, \tau \in 2^{\mathbb{N}}$ *satisfy* $\sigma|_\ell = \tau|_\ell$*, then* $\|x\|_{\mathcal{M}_{\mathfrak{T}(\sigma)}} = \|x\|_{\mathcal{M}_{\mathfrak{T}(\tau)}}$ *for every* $x \in \mathrm{span}\{e_1, e_2, \dots, e_\ell\}$*.*

Definition

For $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$, let $E_{\mathfrak{T}}$ be the completion of $c_{00}(2^{<\mathbb{N}} \setminus \{\emptyset\})$ with the norm

$$\|x\|_{\mathfrak{T}} = \sup_{\sigma \in 2^{\mathbb{N}}} \Big\| \sum_{\ell=1}^{\infty} x(\sigma|_\ell) e_\ell \Big\|_{\mathcal{M}_{\mathfrak{T}(\sigma)}}.$$

Fact

The space $E_{\mathfrak{T}}$ *contains* $T^*[\mathcal{M}_{\mathfrak{T}(\sigma)}, \frac{1}{2}]$ *for every* $\sigma \in 2^{\mathbb{N}}$*.*
Its dual $E_{\mathfrak{T}}^*$ *contains the dual of* $T^*[\mathcal{M}_{\mathfrak{T}(\sigma)}, \frac{1}{2}]$ *for every* $\sigma \in 2^{\mathbb{N}}$*.*

Fact

If $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$ *and* $\sigma, \tau \in 2^{\mathbb{N}}$ *satisfy* $\sigma|_\ell = \tau|_\ell$, *then* $\|x\|_{\mathcal{M}_{\mathfrak{T}(\sigma)}} = \|x\|_{\mathcal{M}_{\mathfrak{T}(\tau)}}$ *for every* $x \in \mathrm{span}\{e_1, e_2, \ldots, e_\ell\}$.

Definition

For $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$, let $E_{\mathfrak{T}}$ be the completion of $c_{00}(2^{<\mathbb{N}} \setminus \{\emptyset\})$ with the norm

$$\|x\|_{\mathfrak{T}} = \sup_{\sigma \in 2^{\mathbb{N}}} \Big\| \sum_{\ell=1}^{\infty} x(\sigma|_\ell) e_\ell \Big\|_{\mathcal{M}_{\mathfrak{T}(\sigma)}}.$$

Fact

The space $E_{\mathfrak{T}}$ *contains* $T^*[\mathcal{M}_{\mathfrak{T}(\sigma)}, \frac{1}{2}]$ *for every* $\sigma \in 2^{\mathbb{N}}$.
Its dual $E_{\mathfrak{T}}^*$ *contains the dual of* $T^*[\mathcal{M}_{\mathfrak{T}(\sigma)}, \frac{1}{2}]$ *for every* $\sigma \in 2^{\mathbb{N}}$.

Remark

The canonical basis $\{e_\eta\}_{\eta \in 2^{<\mathbb{N}} \setminus \{\emptyset\}}$ of $E_{\mathfrak{T}}$ is shrinking.

Proposition

Let $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$.
(1) *If* $\forall \sigma \in 2^{\mathbb{N}} : \mathfrak{T}(\sigma) \in \mathrm{IF}$, *then* $E^*_{\mathfrak{T}}$ *has the Schur and the Dunford-Pettis property.*
(2) *In the opposite case,* $E^*_{\mathfrak{T}}$ *contains a complemented infinite-dimensional reflexive subspace. Thus,* $E^*_{\mathfrak{T}}$ *does not have the Schur nor the Dunford-Pettis property.*

Proposition

Let $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$.
(1) *If* $\forall \sigma \in 2^{\mathbb{N}} : \mathfrak{T}(\sigma) \in \mathrm{IF}$, *then* $E_{\mathfrak{T}}^{*}$ *has the Schur and the Dunford-Pettis property.*
(2) *In the opposite case,* $E_{\mathfrak{T}}^{*}$ *contains a complemented infinite-dimensional reflexive subspace. Thus,* $E_{\mathfrak{T}}^{*}$ *does not have the Schur nor the Dunford-Pettis property.*

Lemma

There exists a Borel mapping $\mathfrak{S} : \mathrm{Tr}(2 \times \mathbb{N}) \to \mathrm{SB}$ *such that* $\mathfrak{S}(\mathfrak{T})$ *is isometric to* $E_{\mathfrak{T}}^{*}$ *for every* $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$.

Proposition

Let $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$.
(1) *If* $\forall \sigma \in 2^{\mathbb{N}} : \mathfrak{T}(\sigma) \in \mathrm{IF}$, *then* $E^*_{\mathfrak{T}}$ *has the Schur and the Dunford-Pettis property.*
(2) *In the opposite case,* $E^*_{\mathfrak{T}}$ *contains a complemented infinite-dimensional reflexive subspace. Thus,* $E^*_{\mathfrak{T}}$ *does not have the Schur nor the Dunford-Pettis property.*

Lemma

There exists a Borel mapping $\mathfrak{S} : \mathrm{Tr}(2 \times \mathbb{N}) \to \mathrm{SB}$ *such that* $\mathfrak{S}(\mathfrak{T})$ *is isometric to* $E^*_{\mathfrak{T}}$ *for every* $\mathfrak{T} \in \mathrm{Tr}(2 \times \mathbb{N})$.

Proof of Theorem 2.

... $\square$

The next step in this line of research is the study of the following problem.

Question (Godefroy)

Is the class of Banach spaces isomorphic to c_0 Borel?

It is possible (and useful) to study the complexity of equivalence relations as well.

Definition

Let E and F be equivalence relations on standard Borel (or Polish) spaces X and Y. We say that E is Borel reducible to F if there is a Borel mapping $f : X \to Y$ such that

$$uEv \Leftrightarrow f(u) F f(v)$$

for all $u, v \in X$.

It is possible (and useful) to study the complexity of equivalence relations as well.

Definition

Let E and F be equivalence relations on standard Borel (or Polish) spaces X and Y. We say that E is Borel reducible to F if there is a Borel mapping $f : X \to Y$ such that

$$uEv \Leftrightarrow f(u)F\,f(v)$$

for all $u, v \in X$.

... then the complexity of E is less than or equal to the complexity of F.

In collaboration with M. Cúth, M. Doucha, J. Grebík and B. Vejnar, we are currently studying relations between equivalence relations like

- E_{GH}: the relation of metric spaces to have Gromov-Hausdorff distance 0,
- $E_{GH,bdd}$: the restriction of E_{GH} to bounded spaces,
- E_K: the relation of Banach spaces to have Kadets distance 0,
- E_{BM}: the relation of Banach spaces to have Banach-Mazur distance 1,
- E_L: the relation of metric spaces to have Lipschitz distance 1.

In collaboration with M. Cúth, M. Doucha, J. Grebík and B. Vejnar, we are currently studying relations between equivalence relations like

- E_{GH}: the relation of metric spaces to have Gromov-Hausdorff distance 0,
- $E_{GH,bdd}$: the restriction of E_{GH} to bounded spaces,
- E_K: the relation of Banach spaces to have Kadets distance 0,
- E_{BM}: the relation of Banach spaces to have Banach-Mazur distance 1,
- E_L: the relation of metric spaces to have Lipschitz distance 1.

Proposition

$E_{GH,bdd}$ is Borel reducible to every equivalence on the list above.