Norming subspaces of Banach spaces

Sebastián Lajara

Departamento de Matemáticas
Universidad de Castilla–La Mancha

Joint work with V. Fonf, S. Troyanski and C. Zanco

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Let E be a Banach space, $X < E$, and $Z < E^*$. TFAE:

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If $\overline{B_Z}^{w^*} = B_{\overline{Z}^{w^*}}$, these conditions are equivalent to:

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Corollary

Let E be a Banach space, $X < E$, $Z < E^*$. If X is reflexive, TFAE:

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5. $E = X \oplus Z_\perp$ and Z is reflexive.

Application to bibasic systems in Banach spaces

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A biorthogonal system $\{x_i, z_i\}_{i=1}^{\infty} \subset E \times E^*$ is said to be bibasic if, $\{x_i\}_i$ is a basic sequence in E, and $\{z_i\}_i$ is a basic sequence in E^*.

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Bon anniversaire, Robert!