Operator ranges and quasi-complemented subspaces of Banach spaces

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Find conditions to ensure the existence of a subspace $L_1 \subset E$ s.t.:

$$L \subset L_1, \qquad L_1 \cap \mathcal{R} = \{0\}, \qquad \text{and} \qquad \operatorname{cl} T(L_1) = \operatorname{cl} T(E).$$

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Definition

Let $A : X \to E$ be a bounded linear operator between Banach spaces. A subspace $L \subset E$ is suitable for A if:

- $L \cap AX = \{0\}$ and
- $\operatorname{codim}(L + AX) = \infty$.

Corollary

Let $A : X \to E$ be a bounded linear operator, E separable. Assume that AX contains a closed subspace E_1 with $\dim(E_1) = \infty$, and let $L \subset E$.

If L is suitable for A then there is a subspace $L_1 \subset E$ such that:

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(i) $L \subset L_1$ and $\operatorname{codim}_{L_1}(L) = \infty$,

(ii) $L_1 \cap AX = \{0\}$, and

(iii) $\operatorname{cl}(L_1 + E_1) = E$.

Corollary

Let E be a separable Banach space and let E_1 and E_2 be subspaces of E such that $E_1 \cap E_2 = \{0\}$ and $\mathrm{cl}\,(E_1 + E_2) = E \neq E_1 + E_2$.

If $A : X \to E_1$ is an operator with $\mathrm{codim}_{E_1}(AX) = \infty$, then there is a subspace $L_1 \subset E_1$ such that:

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Let $A : X \to H$ be a bounded linear operator, H Hilbert separable, and let H_0 be a subspace of H.

If H_0 is suitable for A, then there exist subspaces $V \subset H$ and $W \subset H$ such that:

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Let $A : X \to E$ be a bounded linear operator, with E separable, and L be a subspace of E.

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(i) $G \supset L$ and $\operatorname{codim}_G(L) = \infty$,

(ii) $G \cap AX = H \cap AX = \{0\}$, and

(iii) $\operatorname{cl}(G + H) = E$.

Lemma

Let $A : X \to E$ be an operator between Banach spaces and let $L \subset E$ be a subspace such that $\operatorname{codim}(L + AX) = \infty$.

Then,

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Then, for each $\varepsilon > 0$ there is $g \in L^{\perp}$ with $\|g\| = 1$ such that

$$\sup g(\operatorname{cl} AB_X) \leq \varepsilon.$$